APPLICATIONS OF THE BAIRE CATEGORY THEOREM

The purpose of this talk is two-fold: we first discuss some old and new results in real analysis which involve the Baire category theorem and secondly we state some problems in real analysis and related fields which might have a chance of being resolved using real analysis techniques. The first section concerns results and problems of topological nature while the second section concerns problem of permutation groups.

1 Results and Problems Related to Topology.

Let us first recall the Baire category theorem.

**Theorem 1.** Let $X$ be a complete metric space.

- No nonempty open set is meager (the countable union of nowhere dense sets).
- Countable intersection of sets open and dense in $X$ is dense in $X$.

We say a set is *comeager* if its complement is meager. A generic element of $X$ has Property $P$ means that $\{x \in X : x$ has property $P\}$ is comeager in $X$.

We have the following classical theorem of Banach and Mazurkiewicz which shows that a generic continuous function has bad differentiability property.

**Theorem 2** (Banach, Mazurkiewicz, 1931). A generic $f \in C[0,1]$ is nowhere differentiable. Moreover, we have that $f'(x) = \infty$ and $f'(x) = -\infty$ for all $x \in [0,1]$.

Bruckner and Garg [4] showed that a generic continuous function has bad properties in the sense of level sets as well.
**Theorem 3** (Bruckner-Garg, 1977). A generic $f \in C(X, I)$ has the property that there is a countable dense set $D \subseteq (\min f, \max f)$ such that

- $f^{-1}(y)$ is a singleton set if $y \in \{\min f, \max f\}$,
- $f^{-1}(y)$ is homeomorphic to a Cantor set when $y \in (\min f, \max f) \setminus D$,
- $f^{-1}(y)$ is homeomorphic to the union of a Cantor set and an isolated point when $y \in D$.

In an arbitrary compact metric space, it is not so easy to discuss differentiability. However, one can show that the Bruckner-Garg type theorem holds. This was done by Buczolich and Darji in [3]. We first introduce some terminology necessary for this. Most of the background information and terminology can be found in [10] and [14].

Suppose that $X$ is a compact metric space and $D$ is a decomposition of $X$ into closed sets. For each open set $U$ in $X$, we define

$$[U] = \{A \in D : A \subseteq U\}.$$ If $\bigcup[U]$ is open in $X$ for each open set $U \subseteq X$, then we say that $D$ is an upper semicontinuous decomposition of $X$. The collection $\{[U] : U \text{ is open in } X\}$ is a basis for a topology on $D$ and the corresponding topology is referred to as the upper semicontinuous topology on $X$ generated by $D$.

**Fact 1.** Suppose that $X$ is a compact metric space and $D$ is an upper semicontinuous decomposition of $X$. Then, $D$ is a compact metric space.

We use $\text{Comp}(X)$ to denote the space of components of $X$. It is an upper semicontinuous decomposition of $X$. The following is a well-known fact.

**Fact 2.** $\text{Comp}(X)$, endowed with the upper semicontinuous topology, is totally disconnected; i.e., the only components of $\text{Comp}(X)$ are singletons.

The following generalization of the Bruckner-Garg theorem was proved by Buczolich and Darji in [3].

**Theorem 4** (Buczolich-Darji, 2005). Let $X$ be a continuum with more than one point. For a generic $f \in C(X, I)$ there is a countable dense set $D \subseteq (\min f(X), \max f(X))$ such that

1. if $y \in \{\min f(X), \max f(X)\}$, then $\text{Comp}(f^{-1}(y))$ is a singleton set,
2. if $y \in D$, then $\text{Comp}(f^{-1}(y))$ is homeomorphic to the Cantor set union an isolated point, and
3. if $y \in (\min f(X), \max f(X)) \setminus D$, then $\text{Comp}(f^{-1}(y))$ is homeomorphic to the Cantor set.

Certain aspects of the result stated above were further generalized by Kato where $I$ was replaced by $n$-manifold. We discuss some of his basic results here and refer the reader to his paper [9] for more details.

Let $X$ be a compact metric space and $M_n$ be an $n$-manifold. For any map $f : X \to M_n$, let

$$
E(f, X) = \{ x \in X | x \text{ is a local extreme point of } f \},
$$

$$
S(f, X) = \{ x \in X | x \text{ is a stable point of } f \}
$$

$$
E(f, M_n) = \{ y \in M_n | y \text{ is a local extreme value of } f \}
$$

$$
S(f, M_n) = \{ y \in M_n | y \text{ is a stable value of } f \}.
$$

**Theorem 5** (Kato, 2007). Let $X$ be an everywhere at least $n$-dimensional compactum and $M_n$ be an $n$-manifold, $n \geq 1$. Then there is a dense $G_\delta$-set $G$ in the space $C(X, M_n)$ such that if $f \in G$, then the following properties hold:

- $E(f, X)$ is a dense $F_\sigma$-set in $X$ and $S(f, X)$ is a dense $G_\delta$-set in $X$,
- $E(f, M_n)$ is a dense $F_\sigma$-set in $f(X)$ such that $E(f, M_n) < n - 1$, and
- $S(f, M_n)$ is a dense set in $f(X)$, and hence $f(X)$ is also everywhere at least $n$-dimensional.

**Problem 6.** Is there a more precise description of the fibers of a generic map from a compact metric space $X$ into $M_n$. More precisely, is there a theorem similar to Theorem 4 for generic maps from a compact metric space into manifolds of higher dimension?

We point out here that Kirchheim [11] obtained results of this type in a slightly different direction.

We now move to another aspect of generic maps. In Theorem 4, global structure of fibers of generic maps is described. What can be said about individual components? Again we introduce some terminology. A good reference source for this section is [16].

Let $M$ be a continuum; i.e., a compact, connected, metric space.

We say that $M$ is indecomposable if every proper subcontinuum of $M$ is nowhere dense in $M$.

$M$ is said to be hereditarily indecomposable if every subcontinuum of $M$ is indecomposable.
For a metric space $X$, we let $C(X)$ be the space of all subcontinua of $X$ endowed with the Hausdorff metric. The following is a classical theorem of Bing.

**Theorem 7** (Bing, 1945). Let $n \geq 2$. A generic element of $M$ of $C(\mathbb{R}^n)$ has the property that $M$ is hereditarily indecomposable.


**Theorem 8** (Krasinkiewicz, Levin, 1996). Let $X$ be a compact metric space. Then, a generic $f \in C(X, I)$ has the property that each of its fibers is a Bing compactum, a compactum with all components hereditarily indecomposable.

A detailed analysis of fiber structure of a generic map from $S^2$, the 2-sphere, into $I$ is given in [3]. Some corollaries are stated below.

**Theorem 9** (Buczolich-Darji, 2005). A generic $f \in C(S^2, I)$ has the property that each component of each fiber of $f$ is either a point, or a hereditarily indecomposable continuum which is figure-eight-like.

**Theorem 10** (Buczolich-Darji, 2005). A generic $f \in C(S^2, I)$ has the property that it has many components of fibers which are pseudoarcs, pseudocircles or Lakes of Wada continuum.

The following problems remain open.

**Problem 11.** Does a generic $f \in C(S^2, I)$ have other types of hereditarily indecomposable continuum as components of fibers other than the ones described above?

**Problem 12.** What happens in the situation when we consider a generic map from $S^3$ into $I$, or, in general from $S^n$ into $I$?

### 2 Results and Problems Related to Permutation Groups.

Let $G$ be a Polish group; i.e., a topological group with a complete separable metric. In 1992 Truss [18] introduced the notion of *genericity* in the context of topological groups. The idea is to interrelate algebra and topology somehow. Following Truss, we say that a Polish group $G$ has a generic element if $G$ has a comeager conjugacy class.

Recall that $g, h \in G$ are conjugate if there exists $x \in G$ such that $h = xgx^{-1}$. For $g \in G$, the conjugacy class of $G$ is simply

$$[g] = \{h \in G : h \text{ is conjugate to } g\}.$$
We use $S_{\infty}$ to denote the group of permutations on $\mathbb{N}$. The topology we use on $S_{\infty}$ is the induced topology of $\mathbb{N}^\mathbb{N}$. The group $S_{\infty}$ is Polish and there is a large literature on many aspects of $S_{\infty}$.

Let $\mathcal{V}$ be a countably infinite set, $\mathcal{C} = \{\lambda_1, \ldots, \lambda_t\}$ for some $t \geq 1$, and $F$ be any function from the two-element subsets of $\mathcal{V}$ into $\mathcal{C}$. Then we say that $R_\mathcal{C} = (\mathcal{V}, \mathcal{C}, F)$ is the \textbf{$\mathcal{C}$-coloured random graph} if the following property is satisfied:

\begin{quote}
if $(U_1, \ldots, U_t)$ is any tuple of disjoint finite subsets of $\mathcal{V}$, then there exists $\beta \in \mathcal{V}$ such that $F\{\alpha, \beta\} = \lambda_i$ for all $\alpha \in U_i$ and for all $1 \leq i \leq t$.
\end{quote}

We refer to the above as the \textit{Alice’s restaurant property}; i.e. you can have whatever you want at Alice’s restaurant, and the element $\beta$ as a witness.

A function $f : R_\mathcal{C} \rightarrow R_\mathcal{C}$ is an \textbf{automorphism} of $R_\mathcal{C}$ if $f$ is a bijection and $F\{f(\alpha), f(\beta)\} = F\{\alpha, \beta\}$ for all $\alpha, \beta \in \mathcal{V}$. We will denote the group of automorphisms of $R_\mathcal{C}$ by $\text{Aut}(R_\mathcal{C})$. We note that this group is a $G_\delta$ subset of $S_\mathcal{V}$ and hence a Polish group itself.

A function $f : \mathbb{Q} \rightarrow \mathbb{Q}$ is an \textbf{automorphism} of $\mathbb{Q}$ if it is a bijection and $f(\alpha) < f(\beta)$ whenever $\alpha, \beta \in \mathbb{Q}$ and $\alpha < \beta$. The group of automorphisms of $\mathbb{Q}$ (with operation the composition of functions) is denoted by $\text{Aut}(\mathbb{Q}, \leq)$. Since $\text{Aut}(\mathbb{Q}, \leq)$ is a closed subgroup of $S_\mathbb{Q}$ with the topology of pointwise convergence with $\mathbb{Q}$ endowed with the discrete topology, it follows that $\text{Aut}(\mathbb{Q}, \leq)$ is a Polish group too.

\textbf{Theorem 13} (Truss, 1992). [18] Each of $S_{\infty}$, $\text{Aut}(R_\mathcal{C})$ and $\text{Aut}(\mathbb{Q}, \leq)$ has a generic element.

In each of the above case, Truss gives necessary and sufficient conditions for two elements of the group to be conjugate.

\textbf{Theorem 14.} (Truss, 1992) A generic permutation of $\sigma \in \text{Aut}(R_\mathcal{C})$, has the property that

- $\sigma$ has no infinite cycle,
- for each $k \in \mathbb{N}$, $\sigma$ has infinitely many cycles of length $k$.

We point out that unlike in $S_{\infty}$ if $\sigma, \tau \in \text{Aut}(R_\mathcal{C})$ they may have the same cycle structure but may not be conjugate to each other. However, Truss gives a characterization which describes when two elements of $\text{Aut}(R_\mathcal{C})$ are conjugate to each other.

Recently, Kechris and Rosendal [12] proved a very general theorem from which it follows that many groups have generics.
**Theorem 15** (Truss, Kechris-Rosendal, Ivanov). *The group of automorphisms of certain countable structures have ample generics.*

In 2003 Akin, Hurley and Kennedy [2] asked if \( \mathcal{H}(2^\omega) \), the group of homeomorphisms of the Cantor space, has ample generics. This was independently answered by Kechris-Rosendal [12] and Akin-Glasner-Weiss [1]. The following problem is inspired by some of the earlier work of Truss.

**Problem 16.** Are there some natural/explicit topological conditions which determine if two elements of \( \mathcal{H}(2^\omega) \) are conjugate?

There is a sizable literature on dense, free subgroups of \( S_\infty \). Using Baire category method, Mitchell and I proved some very general theorems from which the following follows:

**Theorem 17** (Mitchell-Darji). [5] [6] Let \( f \in \text{Aut}(\mathbb{R}) \) be a nonidentity element. Then, there is \( g \in \text{Aut}(\mathbb{R}) \) such that \( \langle f, g \rangle \) generate a dense subgroup of \( \text{Aut}(\mathbb{R}) \). Moreover, the less the finite structure of larger the number of \( g \)'s with this property.

**Problem 18.** Are there some conditions which one can place on a certain countable structure so that given any nonidentity automorphism on the structure, one can always find another automorphism on the structure so that these two automorphisms together generate a dense subgroup of the full automorphism group?

We now deviate from the Baire category and discuss coHaar null sets.

**Theorem 19** (Daugherty-Mycielski, 1994). [8] The following sets are Haar null.

- \( \{ \sigma \in S_\infty : \sigma \text{ has infinitely many finite cycles.} \} \)
- \( \{ \sigma \in S_\infty : \sigma \text{ has only finitely many infinite cycle.} \} \)

**Problem 20.** Very little is known as far as Haar null aspects of things. For example:

- What is the cycle structure of an element chosen randomly from \( \text{Aut}(\mathbb{R}) \), \( \text{Aut}(\mathbb{Q}, \leq) \)?
- Which groups have a coHaar null conjugacy class? \( S_\infty \) is not one of them.
References


