ANALOGS OF YOUNG’S CHARACTERIZATION OF BAIRE ONE, DARBOUX FUNCTIONS

In 1907 W.H. Young provided an insightful characterization of those Baire one, real-valued functions of a real variable which have the Darboux property. In this presentation we provide analogs of this characterization for Baire one, real-valued functions of several real variables which have various Darboux-like properties. This report summarizes results presented in two joint works ([2] and [3]) with Paul D. Humke.

1 Introduction.

Of the numerous (see e.g. [1]) characterizations of Baire one functions from \(\mathbb{R}\) to \(\mathbb{R}\) which have the Darboux (i.e., intermediate value) property, one of the most insightful is the following 1907 result of W. H. Young [9].

**Theorem 1.1.** Let \(f: \mathbb{R} \to \mathbb{R}\) belong to Baire class one. Then \(f\) has the Darboux property if and only if \(f\) is bilaterally approachable.

Here the phrase “bilaterally approachable” means that for each \(x \in \mathbb{R}\), there exist sequences \(x_n \uparrow x\) and \(x^*_n \downarrow x\) such that \(f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(x^*_n)\).

There are many ways to extend the notion of “Darboux-ness” to Baire one, real-valued functions defined on \(\mathbb{R}^n\). Often this is done by specifying a class of connected sets in \(\mathbb{R}^n\) which the function must map to intervals. Some of these extensions are compared and contrasted in [4]. In this report we choose to focus on those notions which are amenable to a characterization involving some sort of “approachability” as in Young’s Theorem. We are particularly interested...
in those types of Darboux-ness which are possessed by partial derivatives of differentiable functions.

More specifically, let $\mathcal{A}$ denote the class of connected subsets of $\mathbb{R}^n$ which are mapped to intervals by all partial derivatives of differentiable functions. As pointed out by J. Malý [6], for $n > 1$, $\mathcal{A}$ does not contain all connected subsets of $\mathbb{R}^n$; in particular, he showed that $\mathcal{A}$ does not contain line segments. In that same paper he proved a theorem, one consequence of which is that $\mathcal{A}$ contains the collection of all closed convex sets having nonempty interior. This improved upon a theorem of L. Mišik [6], which showed that a partial derivative of a differentiable function has the property that if that partial assumes two different values on the boundary of a ball, then it assumes every value strictly between those two on the interior of the ball. See C. E. Weil [8] for a concise proof of this fact.

The goal of [2] and [3] was to investigate which of these and similar classes of Baire one, Darboux-like functions are amenable to a Young-type characterization involving an appropriate type of approachability. We shall summarize the results of this investigation.

2 Young-type Characterizations for the Classes of Malý and Mišik.

Let $BC$ denote the class of all Baire one functions from $\mathbb{R}^n \to \mathbb{R}$ which preserve the connectivity of closed balls. It is an easy exercise to see that $BC$ is the class of Baire one functions which satisfy the aforementioned property of Mišik. Also, let $CC$ denote the class of all Baire one functions from $\mathbb{R}^n \to \mathbb{R}$ which preserve the connectivity of each closed convex set having nonempty interior.

We first obtained an analog of Young’s theorem for the class $CC$ by introducing the following definition in [2].

**Definition 2.1.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is **conically approachable at a point** $x \in \mathbb{R}^n$ if for each open cone $S$ with vertex at $x$ there is a sequence of points $\{z_n\}$ in $S$ converging to $x$ such that $\{f(z_n)\}$ converges to $f(x)$. We say that $f$ is **conically approachable** if it is conically approachable at each point in $\mathbb{R}^n$. We use $CA$ to denote the class of conically approachable Baire one functions.

**Theorem 2.2.** Let $f : \mathbb{R}^n \to \mathbb{R}$ belong to Baire class one. Then $f$ maps every closed convex set having nonempty interior to an interval if and only if $f$ is conically approachable; that is, $CC = CA$.

Subsequently, we obtained an analog of Young’s theorem for the class $BC$ by utilizing the following type of approachability in [3].
Definition 2.3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is ball approachable at a point $x \in \mathbb{R}^n$ if for each open ball $S$ having $x$ on its boundary there is a sequence of points $\{z_n\}$ in $S$ converging to $x$ such that $\{f(z_n)\}$ converges to $f(x)$. We say that $f$ is ball approachable if it is ball approachable at each point in $\mathbb{R}^n$. We use $BA$ to denote the class of ball approachable Baire one functions.

Theorem 2.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ belong to Baire class one. Then $f$ maps every closed ball to an interval if and only if $f$ is ball approachable; that is, $BC = BA$.

3 The General Situation.

Having had success in obtaining Young analogs for $BC$ and $CC$, we became interested in pursuing a general result which would not only yield those two characterizations as special cases but also allow us to readily obtain Young analogs for other classes of Baire one, Darboux-like functions. For example, P. Holický, C. E. Weil, and L. Zajíček [5] have recently established a result which implies that the class $A$, noted in the introduction, is notably larger than the class of closed convex sets having nonempty interior. We shall not state their result here, nor shall we give the technical definitions that enabled us to obtain our sought-after general Young analog, but will refer the interested reader to [3].

References


