Math 128E Dec.10, 2004

Final Exam Procedure:

You may use only your calculator, and two 5×8 cards of notes.

Review Topics:

3.4, 4.2, 5.1-5.7, 6.1-6.2, 7.1-7.2, 7.4-7.5, 8.1, 9.1-9.2, 10.1-10.2, 11.1-11.6, M.1-M.4, M.6

Preliminaries

1. Understand inverse trigonometric functions and know derivatives of arcsin and arctan
2. Use of L'Hopital's rule to evaluate limits of indeterminate forms
3. Use of "Σ" notation

Integration

1. **Major Concepts:**
   (a) Interpretation of a definite integral as signed area
   (b) Definition of integral as a limit of Riemann Sums (p.353)
   (c) Fundamental Theorem of Calculus (FTC), both versions (p.322, 325)
   (d) Understand the geometric reasoning used to verify Version 1 of the FTC
   (e) Properties of definite integrals (p.306, 307, 311)
2. Evaluation of definite integrals
   - By exact methods using the FTC and
     (a) recognizing antiderivatives
     (b) using substitution, parts, and calculator.
   - By approximation
     (a) using numerical methods $L_n,\ R_n,\ M_n,\ T_n$
     (b) using derivative properties to determine the accuracy and whether approximations over/under estimate exact value.
3. Applications of Integrals
   For each application below, be able to (1) explain how the application leads to an integral and (2) set up and evaluate the appropriate integral.
   (a) Accumulated change from rate of change, e.g., position from velocity
   (b) Average value for a function
   (c) Arc length of curves
   (d) Area between curves, Volume of solids
   (e) Solving separable differential equations
   (f) Use and interpretation of equations for present value, logistic growth, exponential growth and cooling problems.
4. Improper integrals.
   (a) Recognize both types
   (b) Use limits and antidifferentiation and the Comparison Theorem to determine if improper integrals are divergent or convergent
   (c) Examples of convergent and divergent improper integrals

Sequences and Series

1. Infinite Sequences
   (a) Terminology: general term, convergence, divergence, limit, upper/lower bounds
   (b) Evaluation using limit theorems and L'Hopital's rule
2. Infinite Series
   (a) Terminology: general term (a_n), partial sum (S_n), nth remainder (R_n), limit or sum (S)
   (b) Definition of convergence and divergence of infinite series
   (c) Testing series with constant terms for convergence/divergence and estimating sums and accuracy of sums for convergent series
      - All series using the formal definition, the general (nth) term test for divergence, and recognition as a geometric series or p-series
      - Series with non-negative terms using comparison, ratio and integral tests
      - Series with some negative terms using absolute convergence and alternating series tests
3. Taylor Polynomials and Power Series
   (a) Know the definition of Taylor polynomials and be able to find a Taylor (Maclaurin) polynomial for a function $f(x)$
   (b) General form of a power series and determination of the radius and (open) interval of convergence of a power series

Functions of Two Variables

1. Elementary Graphing of Functions z= f(x,y) in 3-D
   - Plot points in 3-D
   - Recognize and graph equations of planes and cylinders
   - Visualize simple surfaces such as spheres, etc.
   - Graph and label level curves of 3-D surfaces in the x-y plane
2. Partial Derivatives
   - Interpret partial derivatives as slopes of curves of intersection of surface with planes.
   - Compute them symbolically, numerically from tables, and graphically from level curves.
   - Find equations of linear, i.e., tangent plane, approximations of surfaces at specific points.
3. Multiple Integrals
   - Evaluate multiple integrals over simple regions in the x-y plane.

Course Overview

1. What have been the major concepts in the course and how are they interrelated?
2. What has been the role of summation in this course? Where has it been used and how?
3. What has been the role of approximation in this course? Where has it been used, and what techniques were used to determine the accuracy of the approximation?

Review Problems

1. The definite integral:

   (a) Give a geometric interpretation of the definite integral, $\int_a^b f(x)dx$. Be sure to sketch a diagram.

   (b) Write out the formal definition of the definite integral in part (a) using a limit and a Riemann sum

   (c) Explain the relation between the definition of part (b) and the interpretation of part (a). Include a diagram and written explanation.

2. Evaluate the following by finding antiderivatives. In cases where you use substitution or parts indicate an appropriate substitution and give the integral which results from the substitution or use of parts.

   (a) $\displaystyle\int_1^8 \frac{(1+\sqrt{(x)})^3}{\sqrt{x}}\,dx$ (b) $\displaystyle\int \frac{arctan\,x}{1+x^2}\,dx$ (c) $\displaystyle\int x\,cos(x)\,dx$ (d) $\displaystyle\int e^x\sqrt{e^x+5}\,dx$

3. Sketch and describe in words both of the following in $\Re^3$

   (a) $3x + 4y + 2z = 12$ (b) $x - z^2 = 2$

4. Maclaurin series and polynomials:

   (a) Verify, using derivatives, that $P_m(x) = \displaystyle\sum_{k=0}^{2} \frac{x^{k+1}}{k!}$ is a Maclaurin polynomial for the function $f(x) = x\,e^x$.

   (b) Find the interval of convergence for the infinite series, $P(x) = \displaystyle\sum_{k=0}^{\infty} \frac{x^{k+1}}{k!}$

5. Let $f(x, y) = y + y\,sin(x)$

   (a) In the window $[-5, 5] \times [-5, 5]$, sketch and label the level curves for $z = -1,\ 0,\ 1$.

   (b) Find the partial derivatives $f_x(0, 2)$ and $f_y(0, 2)$

   (c) Find the linear approximation, $L(x, y)$ to $f(x, y)$ at $(0, 2)$

6. Let $\displaystyle\int_1^{\infty} \frac{dx}{x^5+\sqrt{x}}$

   (a) Show that this integral converges.

   (b) Find a value of b so that $\displaystyle\int_1^{b} \frac{dx}{x^5+\sqrt{x}}$ approximates the improper integral I with error less than $.0005$.

   (c) For the value of b you determined in part (b), determine which of the numerical approximations L_{10}, M_{10}, R_{10} and T_{10} overestimate $\displaystyle\int_1^{b} \frac{dx}{x^5+\sqrt{x}}$ and which underestimate this integral. Give a geometric (including words and diagrams) explanation for your answer.

   (d) Which of the numerical approximations mentioned in part (c) would you expect to be most accurate? Explain your reasoning.

   (e) What does your result in part (a) tell you about the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^5+\sqrt{k}}$. Explain.

   (f) What is an upper bound for the series in part (e)?

7. Determine if the series $\displaystyle\sum_{k=2}^{\infty} (-1)^{k+1}\frac{ln\,k}{k}$ converges or diverges. Show your work and explain your answer.

8. Show that $y = \dfrac{1}{x+C}$ is a solution of the differential equation $y' = -y^2$. Then find a solution (with a specific value of C) for the initial value problem $y' = -y^2$, $y(0) = .5$.

9. Let R be the region enclosed by $y = 6x - x^2$ and $y = x^2 - 2x$.
   (a) Sketch and shade the region R.
   (b) Set up and evaluate the integral(s) that give the area of the region R.
   (c) Set up and evaluate the integral(s) that give the volume of the solid created when region R is rotated around the line $y = -1$.

10. Let $a_n = \dfrac{n^2}{8n^2 + 6n}$
    (a) Use L'Hôpital's Rule to evaluate $\lim_{n\to\infty} a_n$
    (b) What, if anything, does your result in part (a) tell you about the convergence or divergence of the infinite sequence $\{a_n\}_{n=1}^{\infty}$?
    (c) What, if anything, does your result in part (a) tell you about the convergence or divergence of the infinite series $\sum_{k=1}^{\infty} a_k$?

11. On Dec.13, 2001 the *Star Tribune* reported:

    The 65-and-older population increased worldwide from 131 million in 1950 to 420 million in 2000, according to a report from the Census Bureau and the National Institute on Aging. In the United States, the 2000 census showed about 12 percent or 35 million of the nation's 281.4 million people, were at least 65.

    (a) Assuming that the projected numbers for 65-and-older worldwide population are based on an *exponential* model, find an equation for this model.
    (b) According to your *exponential* model when would the size of the 65-and-older worldwide population be 800 million?
    (c) What would your *exponential* model predict for the size of the 65-and-older United States population in 2055 when you will be part of that group?
    (d) Assuming that the projected numbers for 65-and-older worldwide population are based on a *logistic* model with the same initial relative rate of growth and a carrying capacity of 20 billion, find an equation for this model.
    (e) What would your *logistic* model predict for the size of the 65-and-older world population in 2055?

12. Describe the role of summation in this course. In your description, name specific mathematical topics where summation was used. For each topic, give a *general* description both of the methods used to find sums as well as of the techniques used to determine the accuracy of the sum.

13. Describe the role of approximation in this course. In your description, name specific mathematical topics where approximation was used. For each topic, explain why approximation is needed and give a *general* description both of the methods used to find approximations as well as of the techniques used to determine the accuracy of the approximation.