Test 3 Review Topics: 4.1-4.6, 4.9, 5.1-5.2

Vector Spaces, Subspaces, Bases and Dimension

1. General overview of vectors spaces with understanding of $\Re^n$, $\mathcal{P}_n(t)$ and $\mathcal{M}_{m\times n}$ as examples.
2. **Definition of subspace (pg. 220)
3. Methods of verifying that a subset of a vector space V is, or is not, a subspace of V.
4. Special subspaces connected with a matrix A, namely $Nul\,A$ and $Col\,A$
5. **Definition of basis (pg. 238)
6. Properties of bases for a vector space V: (1) as smallest spanning set for V; (2) as largest linearly independent set in V; (3) each basis for a finite-dimensional V has the same number of vectors.
7. Standard Bases for $\Re^n$, $P_n(x)$, $\mathcal{M}_{m\times n}$
8. Finding bases and dimension of a vector space, including the matrix subspaces, $Col\,A$ and $Nul\,A$.
9. Finding a basis for V
   - by deleting vectors from a spanning set for V.
   - by extending a linearly independent set of vectors in V.
10. Unique representation of a vector in a vector space as a linear combination of basis vectors
11. Finding the coordinate vector $\mathbf{x}_B$ for a vector $\mathbf{x}$ relative to a basis B; use of change-of-coordinate matrix P_B
12. Understanding of coordinate mapping from vector space V to $\Re^n$ as an isomorphism between V and $\Re^n$
13. **Definition of dimension of a vector space, rank of a matrix (pgs. 257, 265)
14. Determination of dimension of $Nul\,A$, $Col\,A$ for a matrix A.
15. Connection between $rank\,A$, $dim\,Col\,A$, and $dim\,Nul\,A$ for a matrix A.
16. **Theorem:** $Rank\,A + dim\,Nul\,A = n$ for an $m \times n$ matrix A.

Linear Transformations and Eigenvectors

1. Definition of isomorphism.
2. Be able to find the kernel and range of a linear transformation.
3. Relation of kernel and range to properties *onto* and *one-to-one.*
4. **Definitions of eigenvectors, eigenvalues (pg. 303)
5. Finding eigenvalues of a square matrix A. Note special case when A is triangular.
6. Finding an eigenspace for a specific eigenvalue
7. Similar matrices have the same eigenvalues.
8. Linear independence of a set of vectors when each vector corresponds to a distinct eigenvalue of a matrix A.

Complete Invertible Matrix Theorem

- See pages 129, 267, 312

Applications to Markov Chains

1. Terminology: probability vector, stochastic matrix, Markov chain, steady-state vector
2. Eigenvalues of stochastic matrices
   - Every stochastic matrix has an eigenvalue $\lambda_1 = 1$
   - If the remaining eigenvalues λ_i, $i \geq 1$, have $|\lambda_i| < 1$, any Markov chain determined by this stochastic matrix converges to an eigenvector corresponding to the eigenvalue 1.

Review Problems:

1. Let $V = span\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}\}$ where $\mathbf{v}_1 = [4, 0, 6, 3]$, $\mathbf{v}_2 = [6, -12, 2, -7]$, $\mathbf{v}_3 = [-1, 6, 2, 5]$, $\mathbf{v}_4 = [-3, 6, -11, -3]$, and $\mathbf{v}_5 = [9, 6, 7, 6]$.

   (a) Find a basis for V. (b) What is the dimension of V?

2. Assume that A is the matrix with columns $\mathbf{v_1}, \ldots, \mathbf{v_5}$ (where the vectors $\mathbf{v_i}$ are those given in problem 1) and T is the linear transformation defined by $T(\mathbf{x}) = \mathbf{Ax}$.
   (a) Then (1) $Col\,A \subset \Re^p$ where $p =$ ____ (2) $Nul\,A \subset \Re^q$ where $q =$ ____
   (b) Find bases for each of the subspaces: (i) $Col\,A$ (ii) $Nul\,A$
   (c) Find integers p, q and r so (1) $Codomain\,T = \Re^p$ (2) $Domain\,T = \Re^q$ (3) $Range\,T \subset \Re^r$
   (d) Find a vector in $ker\,T$ and a vector in $range\,T$.
   (e) Is T an *onto* mapping? Is T a *one-to-one* mapping? Explain.

3. If A is a 7×6 matrix with rank 4, answer each of the following:
   (a) $dim\,Col\,A =$ ____ $dim\,Nul\,A =$ ____
   (b) If $\mathbf{b}$ is a vector in $\Re^7$, will there be solutions for $A\mathbf{x} = \mathbf{b}$? How many? Does it depend on $\mathbf{b}$?

4. Let the linear transformation $T : \mathcal{P}_2 \to \Re^2$ be defined by $T(at^2 + bt + c) = (a + 2b, c - a)$ and let $B = \{1, t, t^2\}$ be a basis for $\mathcal{P}_2$ and let S be the standard basis for $\Re^2$.
   (a) Find the matrix of T relative to bases B and S.
   (b) Use your matrix to find $T(3t^2 - 10)$.

5. Let T be the transformation $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and let $\mathbf{v} = (3, -5)$
   (a) Find $T(\mathbf{v})$
   (b) Find a basis $\mathcal{B}$ for $\Re^2$ that consists of eigenvectors of A.
   (c) Find the change-of-basis matrix $P_{\mathcal{B}}$ for this basis and use it to find $[\mathbf{v}]_{\mathcal{B}}$, where $\mathbf{v}$ is the vector given above.

6. Beginning with the definition of eigenvectors and eigenvalues, explain why the eigenvalues of a matrix A can be found using the equation $det(A - \lambda I) = 0$

7. Let $Q = \begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}$ (a) Verify that Q is stochastic.
   (b) Find 3 linearly independent eigenvectors $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{b_3}$ of Q.
   (c) If the vector $\mathbf{b_1}$ corresponds to the eigenvalue 1, explain why $\mathbf{b_1}$ satisfies the definition of "steady-state vector of Q."
   (d) Show that any vector $\mathbf{x}$ can be written $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$
   (e) Use the above representation of vector $\mathbf{x}$ to show that the product $Q^n\mathbf{x} \to \mathbf{c_1 b_1}$ as $n \to \infty$. What properties of the eigenvalues of Q are essential to your argument?
   (f) Using the information you have found above, what can you say about Markov chains with stage-matrix Q?