Math 220A December 10, 2004

Review Topics: See 1.1-1.10, 2.1-2.3, 3.1-3.2, 4.1-4.6, 4.9, 5.1-5.4, 6.1-6.2

A. **Systems of Linear Equations**

1. Terms: Linear Equation, Consistent, Inconsistent, Homogeneous, Nonhomogeneous, Pivots, Basic and Free Variables, Solution to such a system, Equivalent systems, Trivial, Nontrivial and Unique solution, General and Particular solution
2. Finding solutions for a system of linear equations using row reduction of the augmented (or coefficient) matrix;
3. Recognizing conditions under which a linear system is consistent; under which the solution is unique; relating the solutions of a nonhomogeneous system to those of the corresponding homogeneous system of linear equations

B. **Matrices**

1. Terms: Matrix, Components, Row equivalent, Similar
2. Operations: Scaler Multiples, Sums, Products and Transposes
3. Determinants of square matrices (via expansion about row or column)
4. Multiplicative inverse of a square matrix via reducing the matrix $[A\ I]$
5. Relation of $det(A)$, $det(A^T)$, and $det(A^{-1})$; $det(AB)$ with $det(A)$ and $det(B)$

C. **Vectors in** R^n

1. Geometric and algebraic definitions and properties of vectors
2. Terms: Row and column vectors, Components, Zero vector, vectors $\mathbf{e_i}$, Linear combination, Span (as a noun and as a verb), Spanning set
3. Operations: Addition, Subtraction, Scalar multiplication
4. Parametric vector equations for lines and planes
5. Linear Independence and Dependence of a Set of Vectors (*Know Definitions)
6. Inner (dot) products— definition, properties, and use to find
   - Length (or norm) of a vector, orthogonality of vectors, angle between vectors
   - Orthogonal projection onto a vector
   - Linear independence of orthogonal sets of vectors

D. **Vector Spaces, Subspaces and Bases**

1. *Definition and properties of vector spaces, and subspaces, closure under addition and scalar multiplication
2. Subspaces $nul\,A$, $col\,A$ and their connection with the kernel and range of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ and with the linear transformation properties of *one-to-one* and *onto*
3. Bases:
   (a) *Definition and characterization as smallest spanning set or largest linearly independent set in a vector space
   (b) Standard bases for $\Re^n$, $\mathcal{P}_n(t)$, $\mathcal{M}_{m\times n}$
   (c) Given vector $\mathbf{x}$ and basis B, find the coordinate vector $[\mathbf{x}]_B$ and the change-of-coordinate matrix P_B
   (d) Finding bases and dimension of a vector space, including the matrix subspaces, $Col\ A$ and $Nul\ A$.
   (e) Definition of dimension of a vector space, rank of a matrix
   (f) Connection between $rank\,A$, $dim\,Col\,A$, $dim\,Nul\,A$

E. **Linear Transformations**

1. *Definition; preserving addition and scalar multiplication
2. Determining if a function is a linear transformation
3. Terms: mapping, image, domain, codomain, kernel, range, one-to-one, onto
4. Finding and using a matrix representation for a linear transformation
5. Converting from the matrix $[T]_B$ to $[T]_C$ where B and C are different bases for the domain/codomain.
6. Correspondence of matrix products with composition of linear transformations

F. **Eigenvectors**

1. *Definitions of eigenvectors, eigenvalues
2. Finding eigenvalues using characteristic equations and eigenvectors using row reduction
3. Determining whether a matrix is diagonalizable and finding the matrices P and D for a diagonalizable matrix

G. **Complete Invertible Matrix Theorem** (See pages 129, 267, 312)

H. **Applications to Markov Chains:**

1. Terms: probability vector, stochastic matrix, Markov chain, steady-state vector
2. Eigenvalues of stochastic matrices and conditions under which a Markov chain converges to an eigenvector

Final Review Problems:

1. Let $\mathbf{y} = (6, 3, -2)$, $\mathbf{u}_1 = (3, 4, 0)$, and $\mathbf{u}_2 = (-4, 3, 0)$,

   (a) Show that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal. (b) Find $||\mathbf{y}||$. (c) Find $cos\angle(\mathbf{y}, \mathbf{u}_1)$

   (d) Find $\hat{\mathbf{y}}$, the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}_1$

2. Assume that $T : \Re^n \to \Re^m$ is defined by $T(\mathbf{x}) = A\mathbf{x}$ where A is given below.

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 2 \\ 1 & 2 & 1 & 1 & 2 & 3 \\ 1 & 2 & 0 & 2 & 1 & 4 \\ 3 & 6 & 1 & 4 & 3 & 9 \end{bmatrix}$$

   (a) What is the dimension of the *domain* of T? ______ What is the dimension of the *codomain of* T? ______ Explain your answers.

   (b) Find a basis for $Nul\ A$ and a basis for $Col\ A$.

   (c) Is the transformation T *one-to-one*? ______ Is the transformation T *onto*? ______ Explain.

   (d) If the matrix A above is actually the augmented matrix for the linear system $C\mathbf{x} = \mathbf{b}$, find the solution(s) to the system.

3. Let $T : \Re^n \to \Re^n$ be a linear transformation with diagonalizable matrix A where A has eigenvalues $\lambda_1, \ldots,$ λ_n. How is $det(A)$ related to the eigenvalues of A?

4. Determine whether $Q = \{a + bt + abt^2\}$ is a vector space.

5. Let A be a 3×3 matrix with orthogonal columns all of length one. Find the matrix product $A^T A$.

6. Let $B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ and assume that $C = AB$ where the third row of matrix A is $\mathbf{a}_3 = (-1, 3, 5)$.

   (a) Find and label as many C entries c_{ij} as possible.

   (b) Find the determinant of B using a cofactor expansion.

   (c) Find B^{-1} using elementary row reduction.

7. Let A be a 5×7 matrix. Explain your answers for each of the following.

   (a) If A is the coefficient matrix of a system of linear equations, how many equations are in the system? How many variables are in the system?

   (b) What is the largest possible rank that matrix A can have?

   (c) Is there more than one solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$? to the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$?

   (d) If the rank of A is 4, what is the dimension of $nul\ A$?

   (e) If A is the matrix of a linear transformation $T : V \to W$, what is the dimension of V? the dimension of W?

8. Let $W = Span\, B$ where $B = \{(2, 0, 1), (-1, 0, 1)\}$ and let $\mathbf{v} = (1, 0, 5)$.

   (a) Explain why W is a subspace and name a vector space of which it is a subspace.

   (b) Show that $\mathbf{v}$ is in W.

   (c) Verify that B is a basis for W.

   (d) Find $[\mathbf{v}]_B$.

   (e) Let $T : W \to W$ be the transformation where $T(a, 0, b) = (a + b, 0, 2a)$.

      i. Show that T is a linear transformation.

      ii. Find the B-matrix of T.

      iii. Use your matrix for T to find $[T(\mathbf{v})]_B$.

9. Let T be the transformation $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and let $\mathbf{v} = (3, -5)$

   (a) Find $T(\mathbf{v})$

   (b) Find a basis B for $\Re^2$ that consists of eigenvectors of A.

   (c) Use a diagonal matrix D to compute $[T(\mathbf{v})]_B$

10. Let the linear transformation $T : \Re^2 \to \Re^2$ be defined by $T(1, 0) = (5, -6), \quad T(0, 1) = (-2, 3)$.

   (a) Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

   (b) If $B = \{(2, 1), (0, 3)\}$, find a matrix product which will give the B-matrix of T, i.e., $[T]_B$.

11. Let $S = \{a + bt + 2at^2\}$

   (a) Show that S is a subspace of the vector space $\mathcal{P}_2(t)$.

   (b) Find a basis $\mathcal{B}$ for S.

   (c) Let $T(a + bt + 2at^2) = (a + b, 2a)$.

      i. Find a matrix for T with respect to your basis $\mathcal{B}$ and the standard basis for $\Re^2$.

      ii. Use your matrix to find $T(4 - 5t + 8t^2)$.