Fixed Points of Bimahonian Generating Functions

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Abstract

McMahon's result that states the *length* and *major index* statistics are equidistributed on the symmetric group S_n. Adin, Brenti and Roichman have defined analogues of those statistics for the hyperoctahedral group, B_n and proved equidistribution theorems. Using combinatorial interpretations of Adin, Brenti and Roichman's statistics *length* and *delent*, we consider specializations of bimahonian generating functions for the alternating group A_n. We define a new delent statistic, del_L, for the alternating subgroup of the hyperoctahedral group, $L_n \subset B_n$ and give a combinatorial proof of the bimahonian generating function $\sum_{\pi \in L_n} q^{l_L(\pi)} t^{del_L(\pi)}$. We also give a combinatorial proof of the specialization of the bimahonian generating function for L_n when $q = t = -1$ which verifies Barcelo, Reiner and Stanton's bicyclic sieving phenomenon.

1 Introduction

machamhons equidistribution and specialization. Cyclic sieving phenomenon.

2 Permutation Statistics for S_n

The symmetric group S_n can be generated by the set of Coxeter generators $\{s_1, s_2, \ldots, s_{n-1}\}$ where $s_i = (i\ i+1)$, i.e. s_i interchanges the elements in positions i and $i+1$. (See [6] for a thorough exposition of Coxeter generators). Every permutation in S_n can be written as a product of the s_i's. The minimum number of generators required to express a permutation π is called the *length* of π and is written $l(\pi)$. The canonical presentation for a permutation π is the expression for π in terms of the minimum number of Coxeter generators. The alternating group A_n is the group of even permutations in S_n, i.e. those permutations which can be written as a product of an even number of s_i's.

For a permutation $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n$, define an *inversion* to be a pair (i, j) such that $i < j$ and $\pi_i > \pi_j$. The *inversion statistic*, $inv(\pi)$, is defined as the total number of inversions in π. It is well-known that for any permutation π in S_n, $l(\pi) = inv(\pi)$.

Regev and Roichman [10] define the *delent* statistic for a permutation π in S_n as the number of times the generator s_1 appears in the canonical presentation of π, written $del_S(\pi)$. They go on to give the following combinatorial interpretation of this statistic. Let $\pi \in S_n$. Then j is a *left-to-right minimum* of π if $\pi_i > \pi_j$ for all $1 \leq i < j$. Let $Del_S(\pi) = \{1 < j \leq n | \ j \text{ is a left-to-right minimum}\}$. Then the *delent* statistic $del_S(\pi) = |Del_S(\pi)|$. For example, for the permutation $\pi = 5\ 4\ 3\ 8\ 2\ 9\ 1\ 7\ 6$, $inv(\pi) = 18$ and $del_S(\pi) = 4$.

Mitsuhashi [9] described a generating set for the alternating group A_n. Let $a_i = s_1 s_{i+1}$ so $a_i = a_i^{-1}$ and $a_1^2 = a_1^{-1}$. As in S_n, for a permutation $\sigma \in A_{n+1}$, the length in A of σ, written $l_A(\sigma)$ is equal to the number of generators in the canonical presentation for σ using the set of generators $\{a_1, a_2, \ldots, a_{n-1}\}$. In addition, $l_A(\sigma) = l_S(\sigma) - del_S(\sigma) = inv(\sigma) - del_S(\sigma)$.

For $\sigma \in A_{n+1}$, Regev and Roichman define the A-delent number of σ, written $del_A(\sigma)$, as the number of times a_1 or a_1^{-1} appears in the canonical presentation for σ. The del_A statistic can be interpreted as the number of *almost left-to-right minima* in σ, where j is an almost left-to-right minimum if $\sigma_i < \sigma_j$ for at most one $i < j$.

Let O_n be the set of odd permutations in S_n. The odd permutations do not form a group, since they do not contain the identity and are not closed. For any element $\omega \in O_n$, ω can be written uniquely as $s_1\sigma$ where σ is an element of A_n. Note that $s_1\omega = s_1 s_1 \sigma = \sigma$. Now define the length in O_n of ω, $l_O(\omega) = l_O(s_1\sigma) = l_A(\sigma)$.

Lemma 1. *For $\omega \in O_n$, $l_O(\omega) = inv(\omega) - del_S(\omega) == l_S(\omega) - del_S(\omega)$*

Proof. By definition, $l_O(\omega) = l_A(\sigma)$ and from the definition of l_A, $l_A(\sigma) = l_S(\sigma) - del_S(\sigma)$. The permutation $s_1\sigma$ differs from σ in that 1 and 2 are interchanged. If 1 appeared to the left of 2 in $\omega = s_1\sigma$, then the number of inversions in σ is one greater than the number of inversions in ω. In addition, the number of left to right minima in σ is one greater than the number in $\omega = s_1\sigma$. Thus

$$
\begin{aligned}
l_O(\omega) &= l_O(s_1\sigma) \\
&= l_A(\sigma) \\
&= l_S(\sigma) - del_S(\sigma) \\
&= inv(\sigma) = del_S(\sigma) \\
&= (inv(s_1\sigma) + 1) - (del_S(s_1\sigma) + 1) \\
&= inv(s_1\sigma) - del_S(s_1\sigma) \\
&= inv(\omega) - del_S(\omega) \\
&= l_S(\omega) - del_S(\omega).
\end{aligned}
$$

If 1 appeared to the right of 2 in ω, then the number of inversions in σ is one less than the number of inversions in $\omega = s_1\sigma$. In addition, the number of left to right minima in σ is one less than the number in ω. Thus

$$
\begin{aligned}
l_O(\omega) &= l_O(s_1\sigma) \\
&= l_A(\sigma) \\
&= l_S(\sigma) - del_S(\sigma) \\
&= inv(\sigma) - del_S(\sigma) \\
&= (inv(s_1\sigma) - 1) - (del_S(s_1\sigma) - 1) \\
&= inv(s_1\sigma) - del_S(s_1\sigma) \\
&= inv(\omega) - del_S(\omega) \\
&= l_S(\omega) - del_S(\omega).
\end{aligned}
$$

□

We can define the delent statistic for O_n in terms of the Coxeter generators for A_n and in terms of almost left to right minima. For $\omega \in O_n$, define $del_O(\omega)$ as $del_A(\sigma)$ where $\omega = s_1\sigma$. Thus $del_O(\omega) = del_O(s_1\sigma)$ is the number of a_1 or a_1^{-1} in $\sigma \in A_n$. In addition, since swapping 1 and 2 in ω does not affect the number of almost left to right minima, we have that $del_O(\omega) = del_A(\sigma) =$ the number of almost left to right minima in σ is equal to the number of almost left to right minima in ω.

3 Bivariate Generating Functions for S_n, A_n and O_n

Regev and Roichman proved the following two theorems:

Theorem 1. *(Regev and Roichman, 2004)*

$$\sum_{\pi \in S_n} q^{l_S(\pi)} t^{del_S(\pi)} = (1 + qt)(1 + q + q^2 t) \cdots (1 + q + q^2 + \cdots + q^{n-1} t).$$

Theorem 2. *(Regev and Roichman, 2004)*

$$\sum_{\sigma \in A_{n+1}} q^{l_A(\sigma)} t^{del_A(\sigma)} = (1+2qt)(1+q+2q^2t)\cdots(1+q+q^2+\cdots+2q^{n-1}t).$$

We now specialize $q = t = -1$ in these two theorems and prove the result.

Theorem 3.

$$\sum_{\pi \in S_{2n}} (-1)^{inv(\pi)}(-1)^{del_S(\pi)} = (-1)^{n-1}2^n$$

and

$$\sum_{\pi \in S_{2n+1}} (-1)^{inv(\pi)}(-1)^{del_S(\pi)} = (-1)^{n}2^n$$

We will first prove the result for $\pi \in S_{2n}$. We will use induction on the size of n and give a sign-reversing involution which will have the appropriate number of fixed points.

For $n = 1$, there are only two permutations in S_2, $\pi_1 = 12$ and $\pi_2 = 21$. Since $inv(\pi_1) = 0$ and $del_S(\pi_1) = 0$, $(-1)^{inv(\pi_1)}(-1)^{del_S(\pi_1)} = 1$. Also, since $inv(\pi_2) = 1$ and $del_S(\pi_2) = 1$, $(-1)^{inv(\pi_2)}(-1)^{del_S(\pi_2)} = 1$, thus the sum is $2 = (-1)^{(1-1)}2^1$.

Now we assume the result is true for n and we will prove it is true for $n+1$. Let $\pi \in S_{2n+2}$. Then the numbers $2n+1$ and $2n+2$ can appear in π in several ways.

Case 1: Suppose $2n+1$ and $2n+2$ are in non-adjacent positions with $2n+1$ appearing before $2n+2$ in π. Thus

$$\pi = \pi_1 \quad \pi_2 \quad \cdots \quad \pi_{i-1} \quad 2n+1 \quad \pi_{i+1} \quad \cdots \quad \pi_{j-1} \quad 2n+2 \quad \pi_{j+1} \quad \cdots \quad \pi_{2n+2}.$$

Now form σ by interchanging $2n+1$ and $2n+2$ in π so

$$\sigma = \pi_1 \quad \cdots \quad \pi_{i-1} \quad 2n+2 \quad \pi_{i+1} \quad \cdots \quad \pi_{j-1} \quad 2n+1 \quad \pi_{j+1} \quad \cdots \quad \pi_{2n+2}.$$

Any inversions in π between the elements 1 through $2n$ also exist as inversions in σ since these elements appear in the same order in both permutations. In π, both $2n+1$ and $2n+2$ form an inversion with each of the elements π_{j+1} through π_{2n+2}. In addition, $2n+1$ forms an inversion with each of the elements π_{i+1} through π_{j-1}. In σ, both $2n+1$ and $2n+2$ form an inversion with each of the elements π_{j+1} through π_{2n+2}. The number $2n+1$ no longer forms inversions with the elements π_{i+1} through π_{j-1}, but $2n+2$ now forms an inversion with each of these elements. There is one additional inversion in σ formed between $2n+1$ and $2n+2$, thus the total number of inversions in σ is one greater than the number of inversions in π.

Since $2n+1$ and $2n+2$ are larger than all other elements in π and σ, they will not affect any left-to-right minima that exist among π_1 through π_{i-1}, π_{i+1} through π_{j-1}, or π_{j+1} through π_{2n+2}. In addition, since $2n+1$ and $2n+2$

are non-adjacent, neither can be a left-to-right minimum so the total number of left-to-right minima in π and σ is the same, thus $del_S(\pi) = del_S(\sigma)$.

Then for permutations in S_{2n+2} of this type, $(-1)^{inv(\pi)}(-1)^{del_S(\pi)}$ and $(-1)^{inv(\sigma)}(-1)^{del_S(\sigma)}$ have the opposite parity, thus they will cancel each other out in the sign reversing involution.

Case 2: Suppose that $2n+1$ and $2n+2$ are in adjacent positions in S_{2n+2}, with $2n+1$ occuring first in the permutation. Then

$$\pi = \pi_1 \quad \cdots \quad \pi_{j-1} \quad 2n+1 \quad 2n+2 \quad \pi_{j+2} \quad \cdots \quad \pi_{2n+2}.$$

Again we will form σ by swapping $2n+1$ and $2n+2$ so

$$\sigma = \pi_1 \quad \cdots \quad \pi_{j-1} \quad 2n+2 \quad 2n+1 \quad \pi_{j+2} \quad \cdots \quad \pi_{2n+2}.$$

Any inversions formed between elements 1 through $2n$ that occur in π also occur in σ since the order of these elements remains the same in both permutations. The elements $2n+1$ and $2n+2$ form inversions with all of the elements π_{j+2} through π_{2n+2} in both π and σ. However, in σ there is one additional inversion between $2n+2$ and $2n+1$ so the total number of inversions in σ is one greater than the number of inversions in π.

If $j \neq 1$, then there is at least one element π_{j-1} that is less than both $2n+1$ and $2n+2$ and to the left of both, so neither $2n+1$ nor $2n+2$ can be a left-to-right minimum. Since $2n+1$ and $2n+2$ are larger than all other elements in π, any left-to-right minima that exist in π also exist in σ. Then in this case, the number of left-to-right minima in π and σ are the same, so $del_S(\pi) = del_S(\sigma)$ and thus $(-1)^{inv(\pi)}(-1)^{del_S(\pi)}$ and $(-1)^{inv(\sigma)}(-1)^{del_S(\sigma)}$ have the opposite parity so they will cancel each other out in the sign reversing involution.

If $j = 1$, then $2n+1$ is a new left-to-right minima in σ that did not exist in π so $del_S(\pi) + 1 = del_S(\sigma)$. Then $(-1)^{inv(\pi)}(-1)^{del_S(\pi)}$ and $(-1)^{inv(\sigma)}(-1)^{del_S(\sigma)} = (-1)^{inv(\pi)+1}(-1)^{del_S(\pi)+1}$ have the same parity.

In this case, let $\omega = \pi_3\pi_4\cdots\pi_{2n+2}$. Note that $\omega \in S_{2n}$ so we can apply the sign-reversing involution to ω by induction. If ω is not a fixed point of the sign-reversing involution, then let $\nu \in S_{2n}$ be the element of opposite parity that is paired with it. Then $\pi = \;\; 2n+1 \quad 2n+2 \quad \omega$ and $\mu = \;\; 2n+1 \quad 2n+2 \quad \nu$ are in S_{2n+2} and have opposite parity and thus will cancel each other out in the sign-reversing involution on S_{2n+2}. In addition, $\sigma = \;\; 2n+2 \quad 2n+1 \quad \omega$ and $\tau = \;\; 2n+2 \quad 2n+1 \quad \nu$ are in S_{2n+2} and also have opposite parity so they will also cancel each other out in the sign-reversing involution.

If ω is a fixed point of the sign-reversing involution on S_{2n}, then both $\pi = \;\; 2n+1 \quad 2n+2 \quad \omega$ and $\sigma = \;\; 2n+2 \quad 2n+1 \quad \omega$ will be fixed points of the sign-reversing involution on S_{2n+2}. Both $2n+1$ and $2n+2$ will form inversions with all of the $2n$ elements in ω so the number of inversions in π will be $2n$ greater than the number of inversions in ω. The number of left-to-right minima in π will be one greater than the number of left-to-right minima in ω since there is a new left-to-right minimum in position 3. Thus π and ω have opposite parity (and therefore σ and ω also have opposite parity since π and σ have the same parity). The number of fixed points of S_{2n} is 2^n by induction with sign $(-1)^{n-1}$ and we have shown that each fixed point in S_{2n} gives rise to two fixed points in S_{2n+2}, of opposite parity as those in S_{2n} so the number of fixed points in S_{2n+2} is $(-1)^{(n-1)}2^n(-1)2 = (-1)^n 2^{(n+1)}$.

The proof for elements in S_{2n+1} is similar and we omit the details.

Theorem 4.

$$\sum_{\pi \in A_{2n}} (-1)^{l_A(\pi)}(-1)^{del_A(\pi)} = (-1)^{n-1}2^{(n-1)}3^{(n-1)}$$

and

$$\sum_{\pi \in A_{2n+1}} (-1)^{l_A(\pi)}(-1)^{del_A(\pi)} = (-1)^{(n-1)}2^{(n-1)}3^{n}$$

Proof. Throughout the proof, we will use that $l_A(\pi) = inv(\pi) - del_S(\pi)$. As in the proof of 1, we will first prove the result for A_{2n} and we will proceed by induction on the size of n. If $n = 1$, there is only one permutation $\pi = 12$ in A_2 and both $l_A(\pi)$ and $del_A(\pi)$ are zero, so $(-1)^{l_A(\pi)}(-1)^{del_A(\pi)} = 1$.

Now we assume the result is true for n and we will prove it is true for $n+1$. Let $\pi \in A_{2n+2}$. Then the numbers $2n+1$ and $2n+2$ can appear in π in several ways.

Case 1: Suppose $2n+1$ and $2n+2$ are in non-adjacent positions with $2n+1$ appearing before $2n+2$ in π and not in position 1. Let ω be the permutation in S_{2n} formed by removing $2n+1$ and $2n+2$ from π. Note that ω is not necessarily in A_{2n}. Then

$$\pi = \omega_1 \quad \omega_2 \quad \cdots \quad \omega_{i-1} \quad 2n+1 \quad \omega_i \quad \cdots \quad \omega_{j-2} \quad 2n+2 \quad \omega_{j-1} \quad \cdots \quad \omega_{2n}.$$

Now let $\tilde{\omega}$ be the permutation in S_{2n} formed by interchanging 1 and 2 in ω, i.e. $\tilde{\omega} = (12)\omega$. Then let

$$\sigma = \tilde{\omega}_1 \quad \tilde{\omega}_2 \quad \cdots \quad \tilde{\omega}_{i-1} \quad 2n+2 \quad \tilde{\omega}_i \quad \cdots \quad \tilde{\omega}_{j-2} \quad 2n+1 \quad \tilde{\omega}_{j-1} \quad \cdots \quad \tilde{\omega}_{2n}.$$

Any inversions in π between the elements 3 through $2n$ also exist as inversions in σ since these elements appear in the same order in both permutations. Since 1 and 2 are smaller than all other elements in both π and σ, the number of inversions they make with elements 3 through $2n+2$ is the same in both π and σ. If 1 came before 2 in π, then there is an additional inversion in σ between 2 and 1 and if 2 came before 1 in π then there is one less inversion in σ since 1 and 2 will be in order.

In π, both $2n+1$ and $2n+2$ form an inversion with each of the elements ω_{j-1} through ω_{2n}. In addition, $2n+1$ forms an inversion with each of the elements ω_i through ω_{j-2}. In σ, both $2n+1$ and $2n+2$ form an inversion with each of the elements $\tilde{\omega}_{j-1}$ through $\tilde{\omega}_{2n}$. The number $2n+1$ no longer forms inversions with the elements $\tilde{\omega}_i$ through $\tilde{\omega}_{j-2}$, but $2n+2$ now forms inversions with each of these elements. There is one additional inversion in σ formed between $2n+1$ and $2n+2$, thus the total number of inversions in σ differs from the number of inversions in π by zero if 2 came before 1 in π and by two if 1 came before 2 in π. This implies that the parity of the inversion statistic is the same in both π and σ. Since the number of inversions in a permutation is the same as the length of the permutation, for any permutation in A_n the number of inversions must be even. Since the parity of the number of inversions is the same in π and σ, if π were an element of A_{2n+2} then σ is also an element of A_{2n+2}.

Now since $2n+1$ and $2n+2$ are larger than all other elements in π and σ, they will not affect any left-to-right minima or any almost left-to-right minima that exist among ω_1 through ω_{i-1}, ω_i through ω_{j-2}, or ω_{j-1} through ω_{2n}. In addition, since $2n+1$ and $2n+2$ are non-adjacent, neither can be a left-to-right minimum or an almost left-to-right minimum. Without loss of generality, assume ω is the permutation with 1 before 2 and let 1 appear in position l and 2 in position m. Since both 1 and 2 are less than ω_1, ω_2, $\cdots$, ω_{l-1}, if 1 is a left-to-right minimum in π, i.e. $l > 1$, then 2 will be a left-to-right minimum in σ. Also if 1 is an almost-left-to-right minimum in π, i.e. $l > 2$, then 2 will be an almost-left-to-right minimum in σ. Since 2 comes after 1 in π, it cannot be a left-to-right minimum in π but 1 is a left-to-right minimum in σ. If 2 is an almost-left-to-right minimum in π then 1 is an almost-left-to-right minimum in σ. Thus the number of almost-left-to-right minima in π is the same as the number in σ so $del_A(\pi) = del_A(\sigma)$. The number of left-to-right minima in π will differ from the number in σ by one, so the parity of $del_S(\pi)$ differs from the parity of $del_S(\sigma)$ and thus the parity of $l_A(\pi) = inv(\pi) - del_S(\pi)$ differs from the parity of $l_A(\sigma) = inv(\sigma) - del_S(\sigma)$.

Thus for permutations in A_{2n+2} of this type, $(-1)^{l_A(\pi)}(-1)^{del_A(\pi)}$ and $(-1)^{l_A(\sigma)}(-1)^{del_A(\sigma)}$ have opposite parity and will cancel each other out in the sign reversing involution.

Case 2: Suppose that $2n+1$ and $2n+2$ are in positions 1 and 3 with $2n+1$ occuring first in the permutation and with ω and $\tilde{\omega}$ as defined in Case 1. Then π looks like

$$2n+1 \quad \omega_1 \quad 2n+2 \quad \omega_2 \quad \cdots \quad \omega_{2n}$$

Now form σ by interchanging $2n+1$ and $2n+2$ and changing ω to $\tilde{\omega}$ so σ looks like

$$2n+2 \quad \tilde{\omega}_1 \quad 2n+1 \quad \cdots \quad \tilde{\omega}_{2n}$$

As in the previous case, the number of inversions between the elements ω_1 through ω_{2n} differs by one from the number of inversions between the elements $\tilde{\omega}_1$ through $\tilde{\omega}_{2n}$. In π, both $2n+1$ and $2n+2$ form inversions with each of the elements ω_2 through ω_{2n}. In addition, $2n+1$ forms an inversion with ω_1 in π and not in σ while $2n+2$ forms an inversion with $\tilde{\omega}_1$ in σ and not in π. There is one additional inversion in σ formed between $2n+1$ and $2n+2$, so the total number of inversions in σ is either the same as the number of inversions in π or differs from the total number of inversion in π by two.

Since $2n+1$ and $2n+2$ are larger than all other elements in π and σ, they will not affect any left-to-right minima or any almost left-to-right minima that already exist among the numbers 1 through $2n$. Since $2n+2$ is larger than ω_1 it will not be a left-to-right minimum in π and since $2n+1$ is larger than $\tilde{\omega}_1$ it will not be a left-to-right minimum in σ. However, $2n+1$ does form an almost left-to-right minimum in σ since it is less than $2n+2$ and greater than only $\tilde{\omega}_1$ to the left of it. As in Case 1, interchanging 1 and 2 does not affect the number of almost-left-to-right minima but changes the number of left-to-right minima by one so the parity of both del_S and del_A changes from π to σ which implies that the parity of $(-1)^{l_A(\pi)}(-1)^{del_A(\pi)}$ and $(-1)^{l_A(\sigma)}(-1)^{del_A(\sigma)}$ is the same.

Since $\tilde{\omega} = (12)\omega$, either $\omega \in A_{2n}$ or $\tilde{\omega} \in A_{2n}$. Without loss of generality, assume $\omega \in A_{2n}$. If ω was not a fixed point of the sign reversing involution on A_{2n}, then let μ be the permutation in A_{2n} of opposite parity which is paired with ω and let $\tilde{\mu} = (12)\mu$. Then in the sign reversing involution on A_{2n+2}, $\pi = 2n{+}1\, \omega_1\, 2n{+}2\, \omega_2 \cdots \omega_{2n}$ and $\nu = 2n{+}1\, \mu_1\, 2n{+}2\, \mu_2 \cdots \mu_{2n}$ will cancel each other out and $\sigma = 2n{+}2\, \tilde{\omega}_1\, 2n{+}1\, \tilde{\omega}_2 \cdots \tilde{\omega}_{2n}$ and $\tau = 2n{+}2\, \tilde{\mu}_1\, 2n{+}1\, \tilde{\mu}_2 \cdots \tilde{mu}_{2n}$ will cancel each other out.

If ω was a fixed point of the sign reversing involution on A_{2n}, then π and σ will be two new fixed points of the sign reversing involution on A_{2n+2}. In π, $2n+1$ and $2n+2$ each form inversions with $\omega_2, \omega_3, \cdots, \omega_{2n}$ and $2n+1$ also forms an inversion with ω_1 so π has $2(2n-1)+1 = 4n-1$ more inversions than ω. As compared to ω, π has an additional left-to-right minima at ω_1 and a new almost left-to-right minima at ω_2. Thus the parity of $(-1)^{l_A(\pi)}(-1)^{del_A(\pi)}$ (which is the same as the parity of $(-1)^{l_A(\sigma)}(-1)^{del_A(\sigma)}$) is different than the parity of $(-1)^{l_A(\omega)}(-1)^{del_A(\omega)}$, thus the two new fixed points of the sign reversing involution on A_{2n+2} have opposite parity as ω.

Case 3: Suppose that $2n+1$ and $2n+2$ are in adjacent positions in A_{2n+2}, with $2n+1$ occuring first in the permutation and ω and $\tilde{\omega}$ as in the previous cases. Then

$$\pi = \omega_1 \quad \cdots \quad \omega_{i-1} \quad 2n+1 \quad 2n+2 \quad \omega_i \quad \cdots \quad \omega_{2n}.$$

Again we will form σ by swapping $2n+1$ and $2n+2$ and changing ω to $\tilde{\omega}$ so

$$\sigma = \tilde{\omega}_1 \quad \cdots \quad \tilde{\omega}_{i-1} \quad 2n+2 \quad 2n+1 \quad \tilde{\omega}_i \quad \cdots \quad \tilde{\omega}_{2n}.$$

As in the previous case, the number of inversions between the elements ω_1 through ω_{2n} differs by one from the number of inversions between the elements $\tilde{\omega}_1$ through $\tilde{\omega}_{2n}$. The elements $2n+1$ and $2n+2$ form inversions with all of the elements ω_i through ω_{2n} in both π with all of the elements $\tilde{\omega}_i$ through $\tilde{\omega}_{2n}$ in σ. However, in σ there is one additional inversion between $2n+2$ and $2n+1$ so the total number of inversions in σ differs from the number of inversions in π by either zero or two.

If $i \neq 1$ and $i \neq 2$, then in π there are at least two elements ω_{i-2} and ω_{i-1} that are less than both $2n{+}1$ and $2n{+}2$ and to the left of both, so neither $2n{+}1$ nor $2n+2$ can be a left-to-right minimum or an almost left-to-right minimum.

Similarly, in σ both $\tilde{\omega}_{i-2}$ and $\tilde{\omega}_{i-1}$ are less than both $2n+1$ and $2n+2$ and are to the left of both, so neither can be a left-to-right minimum or an almost-left-to-right minimum. Since $2n+1$ and $2n+2$ are larger than all other elements, any left-to-right minima or almost left-to-right minima that exist in π also exist in σ except that interchanging 1 and 2 changes the number of left-to-right minima by one. Then in this case, the number of left-to-right minima in π and σ changes by one and the number of almost left-to-right minima in π and σ is the same. Thus $(-1)^{l_A(\pi)}(-1)^{del_A(\pi)}$ and $(-1)^{l_A(\sigma)}(-1)^{del_A(\sigma)}$ have the opposite parity so they will cancel each other out in the sign reversing involution.

If $j = 1$, then there is a new left-to-right minimum in both π and σ in position 3. In addition, $2n+1$ is a new left-to-right minima in σ that did not exist in π. Since $2n+1$ and $2n+2$ are larger than all other elements in π, any other left-to-right minima or almost left-to-right minima that exist in π also exist in σ except that interchanging 1 and 2 changes the number of left-to-right minima by one. Thus $(-1)^{l_A(\pi)}(-1)^{del_A(\pi)}$ and $(-1)^{l_A(\sigma)}(-1)^{del_A(\sigma)}$ have the same parity.

As in Case 2, either $\omega \in A_{2n}$ or $\tilde{\omega} \in A_{2n}$. Without loss of generality, assume $\omega \in A_{2n}$. Then we can apply the sign-reversing involution to ω by induction. If ω is not a fixed point of the sign-reversing involution, then let $\nu \in A_{2n}$ be the element of opposite parity that is paired with it. Then $\pi = \quad 2n+1 \quad 2n+2 \quad \omega$ and $\mu = \quad 2n+1 \quad 2n+2 \quad \nu$ are in A_{2n+2} and have opposite parity and thus will cancel each other out in the sign-reversing involution on A_{2n+2}. In addition, $\sigma = \quad 2n+2 \quad 2n+1 \quad \tilde{\omega}$ and $\tau = \quad 2n+2 \quad 2n+1 \quad \tilde{\nu}$, where $\tilde{\nu} = (12)\nu$, are in A_{2n+2} and have opposite parity so they will also cancel each other out in the sign-reversing involution.

If ω is a fixed point of the sign-reversing involution on A_{2n}, then both $\pi = \quad 2n+1 \quad 2n+2 \quad \omega$ and $\sigma = \quad 2n+2 \quad 2n+1 \quad \tilde{\omega}$ will be fixed points of the sign-reversing involution on A_{2n+2}. Both $2n+1$ and $2n+2$ will form inversions with all of the $2n$ elements in ω so the number of inversions in π will be $2n$ greater than the number of inversions in ω. The number of left-to-right minima in π will be one greater than the number of left-to-right minima since there is a new left-to-right minimum in position 3. Thus π and ω have opposite parity (and therefore σ and $\tilde{\omega}$ also have opposite parity since π and σ have the same parity). We have shown that in this case, each fixed point in A_{2n} gives rise to two fixed points in A_{2n+2}, of opposite parity.

If $j = 2$ then if ω_2 was a left-to-right minima in ω it remains a left-to-right minimum in π and $\tilde{\omega}_2$ is a left-to-right minimum in σ. In π, ω_2 is less than both $2n+1$ and $2n+2$ and whether or not $\omega_1 > \omega_2$ or $\omega_1 < \omega_2$, it forms a new almost left-to-right minimum in π and similarly $\tilde{\omega}_2$ forms an almost left-to-right minimum in σ. In addition, $2n+1$ is a new almost left-to-right minima in σ that did not exist in π and interchanging 1 and 2 also changes the number of left-to-right minima by one so $del_A(\sigma)$ has the same parity as $del_A(\pi)$. We have already shown that $inv(\sigma)$ has the same parity as $inv(\pi)$ so $(-1)^{l_A(\pi)}(-1)^{del_A(\pi)}$ and $(-1)^{l_A(\sigma)}(-1)^{del_A(\sigma)}$ have the same parity.

As before, either $\omega \in A_{2n}$ or $\tilde{\omega} \in A_{2n}$. Without loss of generality, assume $\omega \in A_{2n}$ so we can apply the sign-reversing involution to ω by induction. If ω is not a fixed point of the sign-reversing involution, then let $\nu \in A_{2n}$ be the element of opposite parity that is paired with ω. Then $\pi = \omega_1 \ 2n+1 \ 2n+2 \ \omega_2 \cdots \omega_{2n}$ and $\mu = \nu_1 \ 2n+1 \ 2n+2 \ \nu_2 \ \cdots \ \nu_{2n}$ are in A_{2n+2} and have opposite parity and thus will cancel each other out in the sign-reversing involution on A_{2n+2}. In addition, $\sigma = \tilde{\omega}_1 \ 2n+2 \ 2n+1 \ \tilde{\omega}_2 \ \cdots \ \tilde{\omega}_{2n}$ and $\tau = \tilde{\nu}_1 \ 2n+2 \ 2n+1 \ \tilde{\nu}_2 \ \cdots \ \tilde{\nu}_{2n}$, where $\tilde{\nu} = (12)\nu$, are in A_{2n+2} and also have opposite parity so they will also cancel each other out in the sign-reversing involution.

If ω is a fixed point of the sign-reversing involution on A_{2n}, then both $\pi = \omega_1 \ 2n+1 \ 2n+2 \ \omega_2 \ \cdots \ \omega_{2n}$ and $\sigma = \tilde{\omega}_1 \ 2n+2 \ 2n+1 \ \tilde{\omega}_2 \ \cdots \ \tilde{\omega}_{2n}$ will be fixed points of the sign-reversing involution on A_{2n+2}. Both $2n+1$ and $2n+2$ will form inversions with all of the $2n$ elements in ω so the number of inversions in π will be $2n$ greater than the number of inversions in ω. The number of left-to-right minima in π will be the same as the number of left-to-right minima in ω. The number of almost left-to-right minima in π will be one greater than the number of almost left-to-right minima since there is a new almost left-to-right minimum in position 4. Thus π and ω have opposite parity (and therefore σ and ω also have opposite parity since π and σ have the same parity). We have shown that in this case, each fixed point in A_{2n} gives rise to two fixed points in A_{2n+2}, of opposite parity.

Finally, since the number of fixed points in A_{2n} is equal to $(-1)^{(n-1)}6^{(n-1)}$, the number of fixed points of A_{2n+2} is equal to $(-1)^{(n-1)}6^{(n-1)}(2+2+2)(-1) = (-1)^n 6^n$.

The proof for elements in A_{2n+1} is similar and we omit the details.

□

Surprisingly, even though O_n is not a group, we obtain the following interesting results about the generating function for the length and delent statistics on this set of permutations.

Theorem 5.

$$\sum_{\sigma \in O_n} q^{l_O(\sigma)} t^{del_O(\sigma)} = \sum_{\sigma \in A_n} q^{l_A(\sigma)} t^{del_A(\sigma)}$$
$$= (1+2qt)(1+q+2q^2t)\cdots(1+q+q^2+\cdots+2q^{n-1}t)$$

Proof. Let $\omega \in O_n$, then $s_1\omega = \sigma \in A_n$. In addition, we can write ω as $s_1\sigma$. As in the proof of Lemma 1, $l_O(\omega) = l_A(\sigma)$. Swapping 1 and 2 in ω does not change the number of almost left to right minima in ω so $del_O(\omega) = del_A(\sigma)$. Thus $q^{l_O(\omega)} t^{del_O(\omega)} = q^{l_A(\sigma)} t^{del_A(\sigma)}$. □

4 Permutation Statistics for B_n

The hyperoctahedral group B_n is the group of all signed permutations of order n. The elements of B_n can be viewed as elements $\pi = \pi_1\pi_2\cdots\pi_n$ of S_n where each π_i can be positive or negative. If π_i is negative then we will denote it by $\overline{\pi_i}$. The set of negative numbers in π is denoted by $Neg(\pi)$ and the number of negative numbers in π is $neg(\pi) = |Neg(\pi)|$. Given the ordering

$$-n < -(n-1) < \cdots < -1 < 1 < 2 < \cdots < n$$

the definition of the descent statistic, the inversion statistic and the major index are the same for B_n as for S_n. However, the usual inversion statistic on B_n is not the same as the length function in terms of the Coxeter generators for type B_n. The Coxeter generators for B_n are the set $\{s_1, s_2, \ldots, s_{n-1}\}$ of generators for S_n plus the addition of s_0 which takes π_1 to $\overline{\pi_1}$ or $\overline{\pi_1}$ to π_1.

Define
$$\overline{inv(\pi)} = \sum_{1 \le j \le n} \sum_{\substack{i<j \\ \overline{\pi_i} > \pi_j}} 1$$

Then the **flag-inversion** statistic for B_n is defined as

$$finv(\pi) = inv(\pi) + \overline{inv(\pi)} + neg(\pi)$$

and is equivalent to the **length** function for π, $l_B(\pi)$, in terms of the Coxeter generators for B_n. The **flag-major index**, given by

$$fmaj(\pi) = 2maj(\pi) + neg(\pi)$$

was introduced by Adin and Roichman [2] who proved $fmaj$ and $finv$ are equidistributed over B_n, thus giving a generalization to B_n of MacMahon's equidistribution theorem for S_n.

For example, for

$$\pi = \begin{array}{ccccccccc} \overline{6} & 1 & 3 & 2 & \overline{9} & 7 & 4 & \overline{8} & \overline{5} \end{array}$$

we have $inv(\pi) = 17$, $\overline{inv(\pi)} = 24$ and $neg(\pi) = 4$ so $finv(\pi) = 45$. In addition, $maj(\pi) = 20$ so $fmaj(\pi) = 2(20) + 4 = 44$. Bernstein [4] gave a description of the **delent** statistic for B_n, del_B, as the number of left-to-right minimum in π which agrees with the definition of this statistic on S_n. Bernstein also gave a generating set for L_n, the subset of B_n consisting of the signed, even permutations. The generating set is the set $\{a_1, a_2, \ldots, a_{n-1}\}$ together with s_0.

The *L-length* of $\pi \in L_n$ is defined as

$$l_L(\pi) := l_B(\pi) - del_B(\pi) = inv(\pi) - del_B(\pi) + \sum_{i \in Neg(\pi^{-1})} i.$$

This function l_L is NOT a length function with respect to any set of generators.

We will define $del_L(\pi)$ as the number of almost left-to-right minima in π. In addition, we will let M_n denote the set of signed odd permutations in B_n. Note that this set of permutations does not form a group. Any element π of M_n can be written as $(a\ b)\sigma$ where a and b are the two smallest elements of π and where σ is an element of L_n. Define the *length* of an element of M_n, l_M, to be $l_M(\pi) = l_L(\sigma)$. By definition, $l_L(\sigma) = inv(\sigma) - del_B(\sigma) + \sum_{i \in Neg(\sigma^{-1})} i$.

Lemma 2. *For $\pi \in M_n$, $l_M(\pi) = inv(\pi) - del_B(\pi) + \sum_{i \in Neg(\pi^{-1})} i$.*

Proof. By definition, $l_M(\pi) = l_L(\sigma)$ and from the definition of l_L, $l_L(\sigma) = inv(\sigma) - del_B(\sigma) + \sum_{i \in Neg(\sigma))^{-1}} i$. The permutation $(a\ b)\sigma$ differs from σ in that a and b are interchanged. If a appeared to the left of b in π, then the number of inversions in σ is one greater than the number of inversions in π. In addition, the number of left to right minima in σ is one greater than the number in $\pi = (a\ b)\sigma$. Thus

$$\begin{aligned} l_M(\pi) &= l_L(\sigma) \\ &= inv(\sigma) - del_B(\sigma) + \sum_{i \in Neg(\sigma^{-1}} i \\ &= (inv(\pi) + 1) - (del_B(\pi) + 1) + \sum_{i \in Neg(\pi)^{-1})} i \\ &= inv(\pi) - del_B(\pi) + \sum_{i \in Neg(\pi^{-1})} i \end{aligned}$$

If a appeared to the right of b in π, then the number of inversions in σ is one less than the number of inversions in π. In addition, the number of left to right minima in σ is one less than the number in π. Thus

$$\begin{aligned} l_M(\pi) &= l_L(\sigma) \\ &= inv(\sigma) - del_B(\sigma) + \sum_{i \in Neg(\sigma^{-1}} i \\ &= (inv(\pi) - 1) - (del_B(\pi) - 1) + \sum_{i \in Neg(\pi)^{-1})} i \\ &= inv(\pi) - del_B(\pi) + \sum_{i \in Neg(\pi^{-1})} i \end{aligned}$$

□

5 Bivariate Generating Functions for B_n, L_n and M_n

We can now give a nice closed formula for the joint distribution of the l_L and del_L statistics.

Theorem 6.

$$\sum_{\sigma \in L_n} q^{l_L(\sigma)} t^{del_L(\sigma)} = (1+q) \cdots (1+q^n)(1+2qt)(1+q+2q^2t) \cdots (1+q+\cdots+2q^{n-2}t)$$

Proof. We will prove the result by induction. It is straightforward to check that for $n = 1$, $n = 2$, and $n = 3$ the result is true. Now we assume the result is true for L_{n-1} and prove that the generating function for L_n is equal to $(1 + q^n)(1 + q + \cdots + q^{n-3} + 2q^{n-2}t)$ times the generating function for L_{n-1}.

Let π be a permutation in L_{n-1}. To create a permutation σ in L_n we will add the element n into the even positions of σ starting from the right. If n is inserted so that there are an even number of elements to the right of it, then *inv* will change by this even amount which means the new permutation will be in L_n. Since n will be the largest element in σ it will not affect any existing left-to-right minima or almost left-to-right minima that exist in π. In addition, since n is unbarred, the addition of n will not affect $\sum_{i \in Neg(\pi^{-1})} i$.

For any $\pi \in L_{n-1}$, the permutation $(x_1\ x_2)\pi$ where x_1 and x_2 are the two smallest elements in π is a permutation in O_{n-1}. To create a permutation σ in L_n we will add the elements n into the odd positions of $(x_1\ x_2)\pi$ starting from the right. If n is inserted into an odd permutation so that there are an odd number of elements to the right of it, then *inv* will change by that odd amount which means the new permutation will be in L_n. Again, since n will be the largest element in σ it will not affect any existing left-to-right minima or almost left-to-right minima that exist in π. In addition, since n is unbarred, the addition of n will not affect $\sum_{i \in Neg(\pi^{-1})} i$.

Thus the permutations in L_n that contain an unbarred n have a generating function that is $(1 + q + q^2 + \cdots + q^{n-3} + 2q^{n-2}t)$ times the generating function for L_{n-1}. Now we show that the contribution to the generating function of those permutations that contain a barred n is $q^n(1 + q + q^2 + \cdots + q^{n-3} + 2q^{n-2}t)$ times the generating function for L_{n-1}.

Let π be an element of L_{n-1} and let $x_1, x_2, \ldots, x_{n-1}$ be the elements that make up π with $x_1 < x_2 < \cdots x_{n-1}$. (For example for the permutation $\pi = \bar{2}13\bar{5}4 \in L_5$, $x_1 = \bar{5}, x_2 = \bar{2}, x_3 = 1, x_4 = 3, x_5 = 4$.) Now replace x_1 in π with $\bar{n}$, replace x_2 with x_1 and so on until replacing x_{n-1} with x_{n-2}. Since the elements in this new word contain elements in the same relative order as in π, the new word has the same length and delent statistics as π. Now create a permutation σ in L_n by inserting the element x_{n-1} into the even positions starting from the right. If x_{n-1} is inserted so that there are an even number of elements to the right of it, then *inv* will change by this even amount which means the new permutation will be in L_n. Since x_{n-1} will be the largest element in σ it will not affect any existing left-to-right minima or almost left-to-right minima that exist in π. In addition, since $\bar{n}$ is really the new addition to the permutation, $\sum_{i \in Neg(\pi^{-1})} i$ changes by n.

For any $\pi \in L_{n-1}$, the permutation $(x_1\ x_2)\pi$ where x_1 and x_2 are the two smallest elements in π is a permutation in O_{n-1}. Again, let $x_1, x_2, \ldots, x_{n-1}$ be the elements that make up $(x_1\ x_2)\pi$ with $x_1 < x_2 < \cdots x_{n-1}$. Now replace x_1 in π with $\bar{n}$, replace x_2 with x_1 and so on until replacing x_{n-1} with x_{n-2}. Since the elements in this new word contain elements in the same relative order as in π, the new word has the same length and delent statistics as $(x_1\ x_2)\pi$. Now create a permutation σ in L_n by inserting the element x_{n-1} into the odd positions starting from the right. If x_{n-1} is inserted so that there are an odd number of elements to the right of it, then *inv* will change by this odd amount which means the new permutation will be in L_n. Since x_{n-1} will be the largest element in σ it will not affect any existing left-to-right minima or almost left-to-right minima that exist in π. In addition, since $\bar{n}$ is really the new addition to the permutation, $\sum_{i\in Neg(\pi^{-1})} i$ changes by n.

Thus the permutations in L_n that contain a barred n have a generating function that is $q^n(1+q+q^2+\cdots+q^{n-3}+2q^{n-2}t)$ times the generating function for L_{n-1}.

Together with those permutations in L_n that contain an unbarred n we have that the generating function for permutations in L_n is $(1+q^n)(1+q+q^2+\cdots+q^{n-3}+2q^{n-2}t)$ times the generating function for L_{n-1}.

□

Once again, even though M_n is not a group, we obtain for following interesting results about the generating function for the length and delent statistics on this set of permutations.

Theorem 7.

$$\sum_{\sigma\in M_n} q^{l_M(\sigma)}t^{del_M(\sigma)} = (1+q)\cdots(1+q^n)(1+2qt)(1+q+2q^2t)\cdots(1+q+\cdots+2q^{n-2}t)$$

Proof. Let $\pi \in M_n$, then $(a\ b)\pi = \sigma \in L_n$, where a and b are the two smallest elements in π. In addition, we can write π as $(a\ b)\sigma$. As in the proof of Lemma 6, $l_M(\omega) = l_L(\sigma)$. Swapping a and b in π does not change the number of almost left to right minima in π so $del_M(\pi) = del_L(\sigma)$. Thus $q^{l_M(\pi)}t^{del_M(\pi)} = q^{l_L(\sigma)}t^{del_L(\sigma)}$ and by Theorem 6 we have the result. □

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