New Generalizations of Rogers-Ramanujan Type Multisum Identities

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Abstract

A generic lemma is given which leads to generalizations of Rogers-Ramanujan type multisum identities of Andrews-Gordon, Bressoud and others. Several such theorems are explicity stated with partial combinatorial interpretations.

1 Introduction

Many of the existing proofs of the Rogers-Ramanujan type identities come from limiting cases of basic hypergeometric series identities or transformations. In particular the notion of a Bailey pair has proved useful in proving multisum generalizations of Rogers-Ramanujan type identities. For a review of the technique of iterating versions of Bailey's lemma to obtain corresponding Rogers-Ramanujan type indentities, see [7].

We will review the motivation for the type of generalization to be discussed by examining a generalization of the classical Rogers-Ramanujan identities in §2. The main lemmas and applications are in §3. Finally, in §4, we will make some combinatorial connection to polynomials with prescribed hook differences, [4].

We use the standard notation for q-series found in [9] and for partitions found in [3]. We will also need the Jacobi triple product identity

$$\sum_{n=-\infty}^{\infty} q^{n^2} x^n = (q^2, -qx, -q/x; q^2)_\infty. \tag{1}$$

2 A Generalization of Rogers-Ramanujan

The Rogers-Ramanujan identities are well known in the theory of partitions. They may be stated analytically as,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q, q^4; q^5)_\infty} \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}. \tag{3}$$

Although it can be easily verified that sums of the form, $\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n}$, do not factor for integer values of m different from $0, 1$, it would be natural to ask if there was some type of product representation for this type of sum. In a recent paper Garrett, Ismail and Stanton showed the following generalization of the classical Rogers-Ramanujan identities. See [8].

Theorem 2.1. *For $m \geq 0$,*

$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q)_n} = \frac{(-1)^m q^{-\binom{m}{2}} a_m(q)}{(q,q^4;q^5)_\infty} + \frac{(-1)^{m+1} q^{-\binom{m}{2}} b_m(q)}{(q^2,q^3;q^5)_\infty}, \tag{4}$$

where

$$a_m(q) = \sum_{\lambda} (-1)^\lambda q^{\lambda(5\lambda-3)/2} \begin{bmatrix} m-1 \\ \lfloor \frac{m+1-5\lambda}{2} \rfloor \end{bmatrix} \tag{5}$$

$$b_m(q) = \sum_{\lambda} (-1)^\lambda q^{\lambda(5\lambda+1)/2} \begin{bmatrix} m-1 \\ \lfloor \frac{m-1-5\lambda}{2} \rfloor \end{bmatrix}. \tag{6}$$

In light of Theorem 2.1, it would be natural to ask if other Rogers-Ramanujan type identities also have similar "m-versions". It turns out that we can write down similar generalizations for many of the well known multisum Rogers-Ramanujan type identities.

3 Main Lemmas

One of the first multisum generalizations of the classical Rogers-Ramanujan identities used the following iterate of a special case of Bailey's lemma.

$$\sum_{s_1 \geq s_2 \cdots \geq s_{k-1} \geq 0} \frac{a^{s_1+\cdots+s_k} q^{s_1^2+\cdots+s_k^2}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-1}}} \tag{7}$$

$$= \frac{1}{(aq;q)_\infty} \sum_{r=0}^{\infty} a^{kr} q^{kr^2} \frac{(1-aq^{2r})(a;q)_r(-1)^r q^{r(r-1)/2}}{(1-a)(q;q)_r}. \tag{8}$$

In particular, when a is replaced by 1 or q the Andrews-Gordon identities for odd modulus result.

Theorem 3.1.

$$\sum_{s_1 \geq s_2 \cdots \geq s_{k-1} \geq 0} \frac{q^{s_1^2+\cdots+s_{k-1}^2}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-1}}} = \frac{(q^{2k+1},q^k,q^{k+1};q^{2k+1})}{(q;q)_\infty}. \tag{9}$$

The analytic form of this identity is originally due to Andrews [1]. The case that $k=2$ gives the first Rogers-Ramanujan identity.

In addition to this multisum result, there are other similar theorems that generalize Rogers-Ramanujan type identities. In a recent paper, Bressoud, Ismail and Stanton gave versions of Bailey's lemma in which the base changed from q to q^2 or q^3. They were able to use these Theorems, along with the standard Bailey's Lemma to obtain many new Rogers-Ramanujan multisum identities. In this section we will prove two lemmas which have, as corollaries, generalizations of several of the new theorems of Bressoud, Ismail and Stanton, as well as generalizations of the Andrews-Gordon and Bressoud identities.

One of the crucial steps in iterating versions of Bailey's Lemma to prove multisum identities is the ability to evaluate the single sum side. Many of the known multisum identities that can be derived from Bailey's lemma rely on the ability to use the Jacobi triple product identity to evaluate the sum in terms of infinite products. Often there is a pair of values for a for which this can be done. As we have seen in the mod 5 case, $a=1$ or $a=q$ seem to be the only values for a for which the single sum factors.

However, the existence of the m-version of the classical Rogers-Ramanujan identities, Theorem 2.1, suggests that rather than trying to find values for which the single sum factors, we should try to write the single sum as a linear combination of the infinite products, perhaps with polynomial coefficients.

Indeed, Ismail showed that this could be done in the odd modulus case when a is replaced by q^{2m} in (7), [10]. Here we will expand on that idea to prove several new m-versions of known multisum theorems. Before proceeding, we will need one more result. The q-Binomial Theorem is a well known summation result which will be used throughout,

Theorem 3.2.

$$(z;q)_n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j z^j q^{\binom{j}{2}}. \tag{10}$$

The main results in this paper are the following lemmas. These lemmas give us a general framework for writing single sums of the type that arise from certain Bailey Lemma iterations as linear combinations of infinite products with polynomial coefficients. The first allows us to evaluate sums when a is replaced by q^{2m} and the second for $a = q^{2m+1}$. It does not appear that these cases can be unified in general.

Lemma 3.3. *For $m \geq 0$, r an integer and p a half-integer,*

$$\begin{aligned} \frac{1}{(q^{2m+1};q)_\infty} & \sum_{n=0}^{\infty} (q^{2m})^{pn} q^{(p+1/2)n^2+(p+1/2-r)n} (-1)^n (1-(q^{2m})^r q^{(2n+1)r}) \frac{(q^{2m+1};q)_n}{(q;q)_n}. \\ = & \sum_{i=1}^{2p+1} a_i(m,r,p) F_p(i), \end{aligned}$$

where,

$$\begin{aligned} a_i(m,r,p) &= (-1)^{i+r+1} q^{m(r-p-mp)+\binom{r+1}{2}+\binom{i}{2}+ir} \sum_t \begin{bmatrix} 2m \\ (2p+1)t+i+r+m \end{bmatrix} \times \\ & \quad (-1)^{2p(t+1)} q^{(2ip+(2p+1)r)t+p(2p+1)t^2}, \end{aligned}$$

and

$$F_p(i) = \frac{(q^{2p+1}, q^i, q^{2p-i+1}; q^{2p+1})_\infty}{(q;q)_\infty}.$$

Proof. To prove the above Lemma multiply top and bottom of the left hand sum, L, by $(q;q)_{2m}$ and do the appropriate cancellation to get,

$$L = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{2mpn+(p+1/2)n^2+(p+1/2-r)n} (q^{n+1};q)_{2m} (1-q^{2m(r)+(2n+1)r}).$$

Expand the $(q^{n+1};q)_{2m}$ term into a single sum over j using the q-Binomial Theorem, (10), then interchange the order of sums to obtain the following,

$$\begin{aligned} L &= \frac{1}{(q;q)_\infty} \sum_{j=0}^{2m} \begin{bmatrix} 2m \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}} \times \\ & \quad \sum_{n=0}^{\infty} (-1)^n q^{2mpn+(p+1/2)n^2+(p+1/2-r)n+nj} (1-q^{r(2m+2n+1)}). \end{aligned}$$

Send n to $n-m$ then split the n-sum into two sums. In the second sum send j to $2m-j$ and n to $-n-1$. Now the powers of q match and we get two sums which are the same over different ranges of n.

$$L = \frac{q^{mr-mp-m^2p+\binom{m}{2}}}{(q;q)_\infty} \sum_{j=0}^{2m} \begin{bmatrix} 2m \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}-jm} \Big(\sum_{n=m}^{\infty} (-1)^{n+m} q^{(p+1/2)n^2+(1/2-m+p-r+j)n} - \sum_{n=-m-1}^{-\infty} (-1)^{n+m-1} q^{(p+1/2)n^2+(1/2-m+p-r+j)n} \Big).$$

At this point we are close to being able to sum the n-sum. We would like to use The Jacobi Triple Product to evaluate this, but the terms from $-m$ to $m-1$ are missing. It turns out that these terms are zero, see proof below, and can be included in order to proceed.

$$L = \frac{q^{mr-mp-m^2p+\binom{m}{2}}}{(q;q)_\infty} \sum_{j=0}^{2m} \begin{bmatrix} 2m \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}-jm} \sum_{n=-\infty}^{\infty} (-1)^{n+m} q^{(p+1/2)n^2+(1/2-m+p-r+j)n}.$$

Now use Jacobi triple product on the n-sum to get the following.

$$L = \frac{q^{mr-mp-m^2p+\binom{m}{2}}}{(q;q)_\infty} \sum_{j=0}^{2m} \begin{bmatrix} 2m \\ j \end{bmatrix} (-1)^{j+m} q^{\binom{j+1}{2}-jm} (q^{2p+1}, q^{2p+j-m-r+1}, q^{m+r-j}; q^{2p+1})_\infty.$$

First, send j to $j+m$ to get,

$$L = \frac{q^{mr-mp-m^2p}}{(q;q)_\infty} \sum_{j=-m}^{m} \begin{bmatrix} 2m \\ j+m \end{bmatrix} (-1)^j q^{\binom{j+1}{2}} (q^{2p+1}, q^{2p+j-r+1}, q^{r-j}; q^{2p+1})_\infty.$$

Now, in order to extract the polynomials from the infinite products we must consider residue class mod $2p+1$, keeping in mind that p may be a half-integer. So, we would like to replace j by $(2p+1)t+i+r$ and sum over t and i in the following way,

$$L = \frac{q^{mr-mp-m^2p}}{(q;q)_\infty} \sum_{i=1}^{2p+1} \sum_{t} \begin{bmatrix} 2m \\ (2p+1)t+i+r+m \end{bmatrix} (-1)^{(2p+1)t+i+r} q^{\binom{(2p+1)t+i+r+1}{2}} \times (q^{2p+1}, q^{2p+(2p+1)t+i+1}, q^{-(2p+1)t-i}; q^{2p+1})_\infty.$$

Now, it is only a matter of multiplying by the appropriate factors to *remove* the t dependence in the infinite products. This introduces a factor of $(-1)^{2p} q^{((2p+1)t+2i)(t+1)/2)}$ and the infinite products are replaced by $F_p(i)$. Provided we address the missing terms, this completes the proof of the Lemma.

We need to address the issue of the missing terms that we added in the proof of the lemma. The terms in question may be written as,

$$\frac{q^{mr-mp-m^2p+\binom{m}{2}}}{(q;q)_\infty} \sum_{j=0}^{2m} \begin{bmatrix} 2m \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}-jm} \sum_{n=-m}^{m-1} (-1)^{n+m} q^{(p+1/2)n^2+(1/2-m+p-r+j)n}.$$

If we interchange the order of the sums and consider the j-sum,

$$\sum_{j=0}^{2m} \begin{bmatrix} 2m \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}-jm+jn}.$$

We can then sum this with the q-Binomial theorem to obtain,

$$\sum_{j=0}^{2m} \begin{bmatrix} 2m \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}-jm+jn} = (q^{n-m+1}, q)_{2m}.$$

We now note that because the n-sum is over a range which forces $-m \leq n \leq m-1$ so that $(q^{n-m+1}, q)_{2m} = 0$. So this calculation justifies the inclusion of the terms in the proof of the lemma. The proof of the lemma is complete.

For technical reasons, we need to state a second lemma which gives a similar result in the case that $a = q^{2m+1}$.

Lemma 3.4. *For $m \geq 0$, r and integer and p a half-integer,*

$$\frac{1}{(q^{2m+2};q)_\infty} \sum_{n=0}^{\infty} (q^{2m+1})^{pn} q^{(p+1/2)n^2+(p+1/2-r)n} (-1)^n (1-(q^{2m+1})^r q^{(2n+1)r}) \frac{(q^{2m+2};q)_n}{(q;q)_n}.$$
$$= \sum_{i=1}^{2p+1} b_i(m,r,p) G_p(i),$$

where,

$$\begin{aligned} b_i(m,r,p) &= (-1)^{i+r+1} q^{-(m+1)((m+1)p-r)+\binom{i+r}{2}} \sum_t \begin{bmatrix} 2m+1 \\ (2p+1)t+i+r+m \end{bmatrix} \times \\ &\qquad (-1)^{2pt} q^{(2ip+(2p+1)r-p)t+p(2p+1)t^2}, \\ G_p(i) &= \frac{(q^{2p+1}, q^{p+i}, q^{p-i+1}; q^{2p+1})_\infty}{(q;q)_\infty}. \end{aligned}$$

The proof is almost identical to the $a = q^{2m}$ case and will be omitted.

In light of the work by Bressoud, Ismail and Stanton on change of base in Bailey pairs, the previous two lemmas give a wide range of generalizations of multisum Rogers-Ramanujan identities. We will highlight some of the more interesting applications and indicate the current scope of the lemmas in proving multisum generalizations.

As the Andrews-Gordon identities are among the most well known Rogers-Ramanujan multisum identities, we begin by stating their m-version.

Theorem 3.5. *If $m \geq 0$, $1 \leq r \leq k$,*

$$\begin{aligned} \sum_{s_1,\cdots,s_{k-1}\geq 0} & \frac{q^{s_1^2+\cdots+s_{k-1}^2+2m(s_1+\cdots+s_{k-1})+s_r+\cdots+s_{k-1}}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-1}}} \\ &= \sum_{i=1}^{2k+1} a_i(m,r,k) F_k(i) \\ &= \sum_{i=1}^{k} (a_i(m,r,k) + a_{2k+1-i}(m,r,k)) F_k(i). \end{aligned}$$

where,

$$\begin{aligned} a_i(m,r,k) &= (-1)^{i+r+1} q^{m(r-k-mk)+\binom{r+1}{2}+\binom{i}{2}+ir} \times \\ &\qquad \sum_t \begin{bmatrix} 2m \\ (2k+1)t+i+r+m \end{bmatrix} q^{(2ik+(2k+1)r)t+(k+2k^2)t^2}. \end{aligned} \tag{11}$$

and,

$$F_k(i) = \frac{(q^{2k+1}, q^i, q^{2k-i+1}; q^{2k+1})_\infty}{(q;q)_\infty}. \tag{12}$$

Proof. The proof follows directly from applying Lemma 3.3 to the single sum side of Equation (7) with $p = k$.

Lemma 3.4 gives the odd case,

Theorem 3.6. *If $m \geq 0$, $1 \leq r \leq k$,*

$$\begin{aligned} \sum_{s_1,\cdots,s_{k-1}\geq 0} & \frac{q^{s_1^2+\cdots+s_{k-1}^2+(2m+1)(s_1+\cdots+s_{k-1})+s_r+\cdots+s_{k-1}}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-1}}} \\ &= \sum_{i=1}^{2k+1} b_i(m,r,k) G_k(i) \\ &= \sum_{i=1}^{k} (b_i(m,r,k) - q^{-(k-i+1)} b_{2k+2-i}(m,r,k)) G_k(i)). \end{aligned}$$

where,

$$\begin{aligned} b_i(m,r,k) &= (-1)^{i+r+1} q^{-(m+1)((m+1)k-r)+\binom{i+r}{2}} \times \\ & \sum_t \begin{bmatrix} 2m+1 \\ (2k+1)t+i+r+m \end{bmatrix} q^{(2ik+(2k+1)r-k)t+(k+2k^2)t^2}. \end{aligned} \tag{13}$$

and,

$$G_k(i) = \frac{(q^{2k+1}, q^{k+i}, q^{k-i+1}; q^{2k+1})_\infty}{(q;q)_\infty}.$$

Proof. Set $p = k$ in Lemma 3.4.

Not only can we obtain generalizations of the Andrews-Gordon identities, but we now know that by using Bressoud, Ismail and Stanton's results, we can generate many new multisum identities with very unusual bases. In their work they prove theorems which give new ways to generate Bailey pairs from old Bailey pairs. They label these identities (S1), (S2), (D1), (D2), (D3), (E1), (E2), (E3), (T1), and (T2). See [7] and [11] for details. It turns out that they are able define a group of words in these symbols such that each word corresponds to an identity of Rogers-Ramanujan type. For example, in their paper, Bressoud et al. use the following limiting case of Bailey's lemma, which they call (D1),

$$\alpha_r'(a,q) = \alpha_r(a^2, q^2), \tag{14}$$

$$\beta_n'(a,q) = \sum_{k=0}^{n} \frac{(-aq;q)_{2k}}{(q^2;q^2)_{n-k}} q^{n-k} \beta_k(a^2, q^2), \tag{15}$$

along with (S1) to prove a theorem which implies the well known Bressoud identities, [6]. The Bressoud identities are similar to the Andrews-Gordon identities but for even modulus.

Theorem 3.7. *(Bressoud) For r and k positive integers with $1 \leq r \leq k$,*

$$\begin{aligned} & \frac{(-a^{1/2}q;q)_\infty}{(aq;q)_\infty} \sum_{s_1,\cdots,s_{k-1}\geq 0} \frac{a^{s_1+\cdots+s_{k-1}} q^{s_1^2+\cdots+s_{k-1}^2+s_r+\cdots+s_{k-1}}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-1}}(-a^{\frac{1}{2}}q;q)_{s_{k-1}}} \\ &= \frac{(-a^{1/2}q;q)_\infty}{(aq;q)_\infty} \sum_{n\geq 0} a^{(k-\frac{1}{2})n} q^{kn^2+(k-r)n} (-1)^n (1 - a^r q^{(2n+1)r}) \frac{(aq;q)_n}{(q;q)_n}. \end{aligned}$$

It becomes clear, upon inspection of the single sum in the above theorem, that Lemma 3.3 can be applied to give an m-generalization of the known multisum identity for even modulus,

Theorem 3.8. *If $m \geq 0$, $1 \leq r \leq k$,*

$$\sum_{s_1,\cdots,s_{k-1}\geq 0} \frac{q^{2m(s_1+\cdots+s_{k-1})+s_1^2+\cdots+s_{k-1}^2+s_r+\cdots+s_{k-1}}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-1}}(-q^{m+1};q)_{s_{k-1}}}$$
$$= \sum_{j=1}^{2k-1} c_j(m,r,k)F_{k-1/2}(j)$$
$$= \Big(\sum_{j=1}^{k-1}(c_j(m,r,k)+c_{2k-j}(m,r,k))F_{k-1/2}(j)\Big) + c_k(m,r,k)F_{k-1/2}(k).$$

where,

$$c_i(m,r,k) = (-1)^i q^{rm-km-km^2+\binom{m+1}{2}} \sum_t \begin{bmatrix} 2m \\ 2kt+r+i+m \end{bmatrix} \times$$
$$(-1)^{r+t+1} q^{\binom{r+i}{2}+r+(-i+2k(r+i))t+(2k^2-k)t^2}.$$

and,

$$F_k(i) = \frac{(q^{2k+1},q^i,q^{2k-i+1};q^{2k+1})_\infty}{(q;q)_\infty},$$

We can also write down the m odd theorem for even modulus by applying Lemma 3.4 but wil omit the details.

In addition to generalizations of the well known Andrews-Gordon and Bressoud identities, we can also derive identities such as the one below.

Theorem 3.9. *For any positive integers integers k and m,*

$$\sum_{s_1,\cdots,s_{k+1}\geq 0} \frac{q^{2m(s_1+s_2)}q^E}{(q;q)_{s_1-s_2}} \prod_{i=3}^{k+2} \frac{(-q^{m2^{i-2}+2^{i-3}};q^{2^{i-3}})_{2s_i}}{(q^{2^{i-2}};q^{2^{i-2}})_{s_{i-1}-s_i}}$$
$$= \frac{1}{(q^{2^k};q^{2^k})_{2m}(q^{2m+1};q)_\infty} \sum_{i=1}^{2^k+4} g_i(m,k)H_i(k),$$

where,

$$g_i(m,k) = (-1)^i q^{m2^k-2m^2} \sum_t \begin{bmatrix} 2m-1 \\ (2^k+4)t+i+m \end{bmatrix}_{q^{2^k}} \times$$
$$q^{2^k\binom{i+1}{2}+2^k(2^k+2+4i)t+2^{k+1}(2^k+4)t^2}, \tag{16}$$

and,

$$H_i(k) = (q^{2^k+4}, q^{2^k i+2}, q^{2^k-2^k i+2}; q^{2^k+4})_\infty, \tag{17}$$

and,

$$E = s_1^2 + s_2^2 + s_2 + s_3 + \cdots + 2^{k-2}s_{k+1} \quad s_{k+2} = 0.$$

When $m = k = 0$, it turns out that this theorem specializes to Rogers-Ramanujan. Although the right hand side seems to have five terms in the sum, four of the terms vanish and the fifth gives the product side of Rogers-Ramanujan.

Because of the extensive scope of this machinery, it would be impossible to list all the identities for which Lemmas 3.3 and 3.4 give generalizations of this type. For example, in a recent paper, [11], Stanton expanded on the work in [7] as explicitly stated the 26 single-sum identities which correspond to the words of length two in their Bailey's Lemmas generators, (S1), (S2), (D1), (D2), (D3), (E1), (E2), (E3), (T1), and (T2). Among the list were several identities known to Slater, including the Rogers-Ramanujan identities, as well as some that seem to be new. Lemmas 3.3 and 3.4 are designed to give m-versions of many of these theorems.

4 Hook Difference Polynomials

We now turn our attention to the polynomial coefficients of the infinite products in Lemmas 3.3 and 3.4. In the mod 5 m-generalization of Rogers-Ramanujan, Theorem 2.1, the polynomial coefficients of the infinite products are well known as generating functions for partitions with difference between parts at least 2 and certain restrictions on the smallest part. We therefore know that these polynomials not only have positive coefficients, but we are lead to a combinatorial understanding of the right side of Theorem 2.1 in terms of partitions as the products have partition interpretations as well.

The discovery of m-generalizations of known multisum theorems leads naturally to an investigation of the polynomials which appear in Theorems 3.5 through 3.9. Particularly, do these polynomials have positive coefficients and if so, do they count something in particular?

It turns out that indeed these polynomials do have positive coefficients and have a known combinatorial interpretation in terms of partitions with prescribed hook differences. In this section we will give the partition interpretation of the theorems that are stated and discuss some of the combinatorial implications.

We must review the definition of the hook difference polynomials as described in [4], but first we need two definitions.

Definition 4.1. Let λ be a partition whose Ferrers graph has a node in the i-th row and j-th column; we call this the (i, j)th node. We define the hook difference at the (i, j)th node to be the number of nodes in the entire ith row of λ minus the number of nodes in the jth column of λ.

We also need to define the diagonals of a partition.

Definition 4.2. We say that the (i, j)th node lies on diagonal c if $i - j = c$.

In light of these definitions we can now define the hook difference polynomials as defined in [4].

Definition 4.3.

$$D_{K,i}(N, M; \alpha, \beta|q) = \sum_{\lambda=-\infty}^{\infty} q^{\lambda(K\lambda+i)(\alpha+\beta)-K\beta\lambda} \begin{bmatrix} N+M \\ N-K\lambda \end{bmatrix}_q - \sum_{\lambda=-\infty}^{\infty} q^{\lambda(K\lambda-i)(\alpha+\beta)-K\beta\lambda+\beta i} \begin{bmatrix} N+M \\ N-K\lambda+i \end{bmatrix}_q .$$

It was shown in [4] that,

Theorem 4.4. *If α and β are positive integers satisfying*

$$\alpha + \beta < K, \quad -i + \beta \le N - M \le K - i - \alpha,$$

and if $1 \le i \le K$, then $D_{K,i}(N, M; \alpha, \beta|q)$ is the generating function for partitions into at most M parts, each part at most N such that the hook differences on diagonal $1-\beta$ are $\ge -i+\beta+1$ and on diagonal $\alpha - 1$ are $\le K - i - \alpha - 1$.

Therefore, the polynomials in question have non-negative coefficients.

We will also note that in the event that α or β is 0, we have the following, $D_{K,i}(N, M; 0, \beta|q)$ is the generating function for partitions subject to the conditions in Theorem 4.4 (with $\alpha = 0$) with the added condition that the number of parts of the partition lies in the closed interval $[N-K+i+1, M]$. And similarly, $D_{K,i}(N, M; \alpha, 0|q)$ is the generating function for partitions as described above with the added conditions that the largest part of the partition lies in the closed interval $[M-i+1, N]$.

We will use this to give an interpretation of the differences of the polynomials in Theorems 3.5 and 3.6 in terms of integer partitions.

First consider the differences of polynomials in Theorem 3.5. Using Definitions 4.1, 4.2 and 4.3, we can rewrite the difference of the a_i's in the following way,

Proposition 4.5. *If $a_i(m, r, k)$ are defined as in Theorem 3.5, then,*

$$(-1)^{i+r} q^A (a_{2k+1-i}(m, r, k) + a_i(m, r, k)) = D_{K,I}(N_1, M_1; \alpha_1, \beta_1|\, q)$$

where,

$$M_1 = m + i - r \quad N_1 = m - i + r \quad K = 2k + 1$$
$$I = 2i \quad \alpha_1 = k - r \quad \beta_1 = r$$
$$A = -\left(\binom{i}{2} + \binom{r+1}{2} - ir + m(r - k - km)\right)$$

We state the analogous property for the b_i's.

Proposition 4.6. *If $b_i(m, r, k)$ are defined as in Theorem 3.6 then,*

$$(-1)^{i+r} q^B (b_i(m, r, k) - q^{-(k-i+1)} b_{2k+2-i}(m, r, k)) = D_{K,I}(N_2, M_2; \alpha_2, \beta_2|\, q)$$

where,

$$\begin{aligned} M_2 &= m + i - r \qquad N_2 = m - i + r + 1 \\ K &= 2k + 1 \qquad I = 2i - 1 \\ \alpha_2 &= k - r \qquad \beta_2 = r \\ B &= -\left(\binom{i}{2} - k - 2km - km^2 + 3r/2 - ir + mr + r^2/2\right) \end{aligned}$$

For completeness, we will restate the m-generalizations of the Andrews-Gordon identities with this combinatorial interpretation.

Corollary 4.7. *(Hook difference interpretation of Andrews-Gordon – even case.) For $m \ge 0$ an integer, $1 \le r \le k$, r and k integers,*

$$\sum_{s_1, \cdots, s_{k-1} \ge 0} \frac{q^{s_1^2 + \cdots + s_{k-1}^2 + 2m(s_1 + \cdots + s_{k-1}) + s_r + \cdots + s_{k-1}}}{(q;q)_{s_1 - s_2} (q;q)_{s_2 - s_3} \cdots (q;q)_{s_{k-1}}}$$
$$= \sum_{i=1}^{k} (-1)^{i+r} q^{-A} D_{K,I}(N_1, M_1; \alpha_1, \beta_1|\, q) F_k(i).$$

where, $F_k(i)$ is defined in Theorem 3.5.

Corollary 4.8. *(Hook difference interpretation of Andrews-Gordon – odd case.) For $m \geq 0$ an integer, $1 \leq r \leq k$, r and k integers,*

$$\sum_{s_1,\cdots,s_{k-1}\geq 0} \frac{q^{s_1^2+\cdots+s_{k-1}^2+(2m+1)(s_1+\cdots+s_{k-1})+s_{k-r+1}+\cdots+s_{k-1}}}{(q;q)_{s_1-s_2}(q;q)_{s_2-s_3}\cdots(q;q)_{s_{k-1}}}$$
$$= \sum_{i=1}^{k}(-1)^{i+r}q^{-B}D_{K,I}(N_2,M_2;\alpha_2,\beta_2|\,q)F_k(i).$$

where, $F_k(i)$ is defined in Theorem 3.5.

Remark: It should be pointed out that the infinite products in the above two corollaries can also be written in terms of hook difference polynomials. It is easy to check that $F_k(i) = \lim_{N,M\to\infty} D_{2k+1,i}(N,M;1,1|\,q)$.

It turns out that we can apply these ideas in a more general setting. In fact, for p an integer we can restate Lemmas 3.3 and 3.4 in terms of hook difference polynomials. We will state below the even case for Lemma 3.3. A similar corollary can be obtained for Lemma 3.4.

Corollary 4.9. *For $m \geq 0$, r and p integers,*

$$\frac{1}{(q^{2m+1};q)_\infty} \sum_{n=0}^{\infty}(q^{2m})^{pn}q^{(p+1/2)n^2+(p+1/2-r)n}(-1)^n(1-(q^{2m})^r q^{(2n+1)r})\frac{(q^{2m+1};q)_n}{(q;q)_n}.$$
$$= \sum_{i=1}^{p}(-1)^{i+r}q^{A}D_{K,I}(N,M;\alpha,\beta|\,q)F_p(i),$$

where,

$$\begin{aligned} N &= r+m-i \qquad M = m+i-r \\ K &= 2p+1 \qquad I = 2i \\ \alpha &= p-r \qquad \beta = r \\ A &= m(r-p-mp)+\binom{r+1}{2}+\binom{i}{2}-ir. \end{aligned}$$

and

$$F_p(i) = \frac{(q^{2p+1},q^i,q^{2p-i+1};q^{2p+1})_\infty}{(q;q)_\infty}.$$

5 Remarks

Although we do not have a complete combinatorial model for the m-generalization of the multisum Rogers-Ramanujan identities presented in this chapter, there seems to be a natural connection between the multisum identities and Hook difference polynomials. More work is needed to understand the combinatorics of these identities.

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