# Weighted Tilings and $q$-Fibonacci Numbers

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November 4, 2004

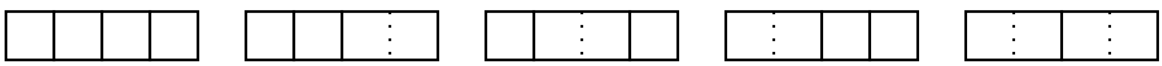
**Abstract**

We exploit recent combinatorial proofs of Fibonacci identities given by Benjamin et. al. which use tilings of an $n \times 1$ board to construct an analogous theory of $q$-Fibonacci identities using weighted tilings. From this combinatorial interpretation, we obtain simple combinatorial proofs of new $q$-analogues of many well known Fibonacci identities.

## 1 Introduction

Benjamin and Quinn have given elegant combinatorial proofs of several known Fibonacci, Lucas, and Gibonacci identitites [2]. The basic counting techniques they used suggest new theorems and new directions for study. We will exploit the power of their methods to give several $q$-analogues of known Fibonacci identities.

The classical Fibonacci numbers have several combinatorial interpretations. We will interpret $f_n := F_{n+1}$ as the number of tilings of an $n$ board with only squares and dominoes. Thus $f_4 = 5$ enumerates the tilings:



We will extend the methods of Benjamin and Quinn by considering a well known $q$-analogue of the Fibonacci numbers. The polynomial generalization of the $f_n$'s given below was first considered by I. Schur [5] in the context of giving a combinatorial proof of the Rogers-Ramanujan identities.

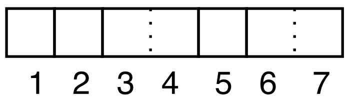
$$f_{n+1}(q) = f_n(q) + q^{n-1} f_{n-1}(q) \qquad f_1(q) = f_2(q) = 1. \tag{1}$$

The polynomials $f_n(q)$ are considered $q$-analogues of the Fibonacci numbers because $\lim_{q\to 1} f_{n+1}(q) = f_n$. For example, $f_5(q) = 1 + q + q^2 + q^3 + q^4$ which clearly tends to $f_4 = 5$ as $q \to 1$. The recurrence (1) suggests we generalize the combinatorial interpretation of Fibonacci numbers by weighting each *domino* with a power of $q$ that corresponds to its location in the tiling. Then

$$f_{n+1}(q) = \sum_{t \in T_n} q^{|t|} \tag{2}$$

where $T_n$ is the set of all tilings of an $n$-board. The weight of a fixed tiling $|t|$ is the sum of all $i$ such that $t$ has a domino in position $(i, i+1)$.

**Example.**




The tiling above has weight $q^3 \cdot q^6 = q^9$ as there is one domino in position $(3, 4)$ and one in position $(6, 7)$.

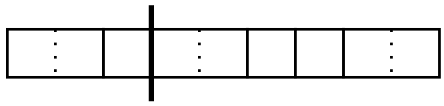
We define a *fault* of a tiling to be a position $m$ such that the tiling does not include a domino in position $(m, m+1)$. Occasionally we will need to split a tiling at a fault and count the possible tilings to the right and to the left of the fault. While tilings to the left of a fault are counted by $q$-Fibonacci numbers, tilings to the right have higher weights. We will define *shifted q-Fibonacci numbers* to count weighted tilings to the right of a fault in a standard weighted tiling as

$$f_{n+1}^{(a)}(q) = f_n^{(a)}(q) + q^{n-1+a} f_{n-1}^{(a)}(q), \qquad f_1^{(a)}(q) = f_2^{(a)}(q) = 1. \tag{3}$$

In particular, $f_{n+1}^{(a)}(q)$ is the generating function for weighted tilings of a board of length $n$ shifted to the right $a$ units. It is easy to verify that $f_{n+1}^{(0)}(q) = f_{n+1}(q)$.

**Example.**

Consider the tiling of a 9-board below. If we break this board at the indicated fault, the weighted tiling on the right side of the fault is given by $q^4 \cdot q^8 = q^{12}$.



If we consider all possible tilings of a 6-board to the right of a fault at the third cell, we would have: $1 + q^4 + q^5 + q^6 + q^7 + q^8 + q^{10} + q^{11} + 2q^{12} + q^{13} + q^{14} + q^{18}$ which is precisely $f_7^{(3)}(q)$.

Lastly, $q$-binomial coefficients play a role in counting weighted tilings. Recall the definition of the $q$-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^k)(1-q)(1-q^2)\cdots(1-q^{n-k})}. \tag{4}$$

Notice that $\lim_{q \to 1^-} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$. It is easy to verify that the $q$-binomial coefficients can be alternately defined in terms of the recurrence

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q, \qquad \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 \text{ and } \begin{bmatrix} n \\ n \end{bmatrix}_q = 1. \tag{5}$$

It is easy to check that $q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q$ counts the number of weighted tilings having $n$ tiles and exactly $k$ dominoes. We show this by letting $g(n,k)$ represent the generating function for weighted tilings with exactly $n$ tiles, $k$ of which are dominoes. Then $g(n,k)$ must satisfy the recurrence

$$g(n,k) = q^{n+k-1} g(n-1,k-1) + g(n-1,k). \tag{6}$$

This is clear if we consider tilings with $n$ tiles, $k$ of which are dominoes, and split them into two disjoint cases depending on the last tile. If the last tile is a domino, the domino contributes $q^{n+k-1}$ to the weight and any weighted tiling using $n-1$ tiles, $k-1$ of them squares, could precede. Thus, we have $q^{n+k-1}g(n-1,k-1)$. If the last tile is a square, then we get a tiling using $n-1$ tiles with exactly $k$ dominoes or $g(n-1,k)$. Multiplying through (6) by $q^{k^2}$ and checking initial conditions gives the desired result. For more information on the $q$-binomial coefficients see Andrews [1].

## 2 Identities

In this section we state several known Fibonacci identities for which combinatorial proofs are known. We prove $q$-analogues using combinatorial methods and weighted tilings.

**Identity 2.1.** *Fibonacci version: For* $n \geq 0$*,*

$$f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1 \tag{7}$$

*$q$-Fibonacci version: For* $n \geq 0$*,*

$$qf_1(q) + q^2 f_2(q) + \cdots + q^{n+1} f_{n+1}(q) = f_{n+3}(q) - 1 \tag{8}$$

*Proof.* Count the number of weighted tilings of an $(n+2)$-board that use at least one domino.

- **RHS:** The tiling of all squares is the only one omitted thus giving $f_{n+3}(q) - 1$.
- **LHS:** Consider the location of the last domino, say position $(k, k+1)$. This domino contributes a $q^k$ to the weight, all tiles to the right are squares and to the left we may have any weighted tiling of a $(k-1)$-board, which gives $f_k(q)$. Summing over all possible $k$ gives the desired result.

**Identity 2.2.** *Fibonacci version: For* $n \geq 0$*,*

$$f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} \tag{9}$$

*q-Fibonacci version: For* $n \geq 0$,

$$q^{n^2+n} f_1(q) + q^{n^2+n-2} f_3(q) + q^{n^2+n-6} f_5(q) \cdots + f_{2n+1}(q) = f_{2n+2}(q) \tag{10}$$

*Proof.* Count the number of weighted tilings of a $(2n+1)$-board.

- **RHS:** $f_{2n+2}(q)$
- **LHS:** Consider the location of the last square. Since the board has odd length there must be such a square and it must occupy an odd-numbered cell. If this cell is in position $k$ then the dominoes to the right of cell $k$ contribute a weight of $q^{(k+1)+(k+3)+\cdots+(2n)} = q^{n^2+n-(k^2-1)/4}$. The remainder is simply a tiling of a $(k-1)$-board given by $f_k(q)$. Summing over all odd $k$ gives the desired result.

**Identity 2.3.** *Fibonacci version: For* $m, n \geq 0$,

$$f_{m+n} = f_m f_n + f_{m-1} f_{n-1} \tag{11}$$

*q-Fibonacci version: For* $m, n \geq 0$,

$$f_{m+n+1}(q) = f_{m+1}(q) f_{n+1}^{(m)}(q) + q^m f_m(q) f_n^{(m+1)}(q) \tag{12}$$

*Proof.* Count the number of weighted tilings of an $(m+n)$-board.

- **RHS:** Consider the disjoint cases where a tiling has or does not have a fault after the $m$th cell. If a weighted tiling has such a fault then to the left of the fault we have a standard weighted tiling of an $m$-board $f_{m+1}(q)$ followed by a *shifted* weighted tiling of an $n$-board, $f_{n+1}^{(m)}(q)$. If there is no fault after the $m$th cell, then a domino must occur in position $(m, m+1)$. That domino contributes $q^m$ to the weight. There is a weighted tiling to the left of an $(m-1)$-board and a shifted weighted tiling of an $(n-1)$-board on the right.
- **LHS:** $f_{m+n+1}(q)$

**Identity 2.4.** *Fibonacci version: For $n \geq 0$,*

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = f_n \tag{13}$$

*q-Fibonacci version: For $n \geq 0$,*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q = f_{n+1}(q) \tag{14}$$

Note: This is often used as the definition of $q$-Fibonacci numbers and is the definition of polynomials considered by Schur [5].

*Proof.* Count the number of weighted tilings of an $n$-board.

- **RHS:** $f_{n+1}(q)$
- **LHS:** Let $k$ be the number of dominoes in the weighted tiling. A tiling of an $n$-board with $k$ dominoes must have $n-2k$ squares. Such a tiling with $n-k$ tiles, exactly $k$ of which are dominoes, is counted by $q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q$. Summing over all possible $k$ gives the desired result.

**Identity 2.5.** *Fibonacci version: For $n \geq 0$,*

$$\sum_{i \geq 0} \sum_{j \geq 0} \binom{n-i}{j} \binom{n-j}{i} = f_{2n+1} \tag{15}$$

*q-Fibonacci version: For $n \geq 0$,*

$$\sum_{i \geq 0} \sum_{j \geq 0} q^{(i^2+(n+i+1)j)} \begin{bmatrix} n-i \\ j \end{bmatrix}_q \begin{bmatrix} n-j \\ i \end{bmatrix}_q = f_{2n+2}(q) \tag{16}$$

*Proof.* Count the number of weighted tilings of a $(2n+1)$-board.

- **RHS:** $f_{2n+2}(q)$
- **LHS:** Consider the median square (there must be an odd number of squares). Count the number of weighted tilings with exactly $i$ dominoes to the left of the median square and $j$ to the right. If a tiling of a $(2n+1)$-board has $i+j$ dominoes, it must

have $2n+1-2i-2j$ squares, $n-i-j$ squares on each side of the median square. So, the tiling to the left of the median square is a standard tiling with exactly $n-j$ tiles $i$ of which are dominoes, thus contributing a factor of $q^{i^2}\begin{bmatrix} n-j \\ i \end{bmatrix}_q$. The contribution to the right of the median square is complicated by the fact that the weights are shifted. It is not hard to check that this part of the tiling is given by $q^{(n+i+1)j}\begin{bmatrix} n-i \\ j \end{bmatrix}_q$. Summing over all possible $i$ and $j$ gives the desired result.

**Identity 2.6.** *Fibonacci version: For $n \geq 0$,*

$$\sum_{k=1}^{n} \binom{n}{k} f_{k-1} = f_{2n-1} \tag{17}$$

*q-Fibonacci version: For $n \geq 0$,*

$$\sum_{k=1}^{n} q^{(n-k)^2} \begin{bmatrix} n \\ n-k \end{bmatrix}_q f_k^{(2n-k)}(q) = f_{2n}(q) \tag{18}$$

*Proof.* Count the number of weighted tilings of a $(2n-1)$-board.

- **RHS:** $f_{2n}(q)$
- **LHS:** Let $k$ be the number of squares among the first $n$ tiles. Then we have a tiling with $n-k$ dominoes, thus $q^{(n-k)^2}\begin{bmatrix} n \\ n-k \end{bmatrix}_q$. The remainder is a shifted weighted tiling of a $(k-1)$-board giving $f_k^{(2n-k)}(q)$.

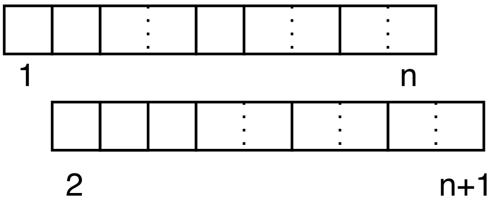
**Identity 2.7.** *Fibonacci version: For $n \geq 0$,*

$$f_n^2 = f_{n+1} f_{n-1} + (-1)^n \tag{19}$$

*q-Fibonacci version: For $n \geq 0$,*

$$f_{n+1}(q) f_{n+1}^{(1)}(q) = f_{n+2}(q) f_n^{(1)}(q) + (-1)^n q^{\binom{n+1}{2}} \tag{20}$$

*Proof.* Consider a pair of tilings of $n$-boards. Draw one below the other shifted to the right one position as in the figure below.



- **RHS:** Consider the last common fault. Break both tilings at that fault and swap the tails of the tilings. This results in a tiling of an $(n+1)$-board on top with a shifted tiling of an $(n-1)$-board below. This is simply $f_{n+2}(q)f_n^{(1)}(q)$. We cannot make such a swap when the two tilings do not share a common fault. This occurs *only* in the case where both tilings are all dominoes. When $n$ is even, the all-domino pair of tilings are counted on the right side and so must be added to the left side, thus contributing $q^{\binom{n+1}{2}}$ to the right side. If $n$ is odd, then the pair of all domino tilings cannot exist on the left side, but the pair of all domino tilings **does** exist on the right side and must be subtracted, thus the $-q^{\binom{n+1}{2}}$.

- **LHS:** The pair of weighted tilings is counted by $f_{n+1}(q)f_{n+1}^{(1)}(q)$.

**Identity 2.8.** *Fibonacci version: For $n \geq 0$,*

$$\sum_{k=0}^{n} f_k^2 = f_n f_{n+1} \tag{21}$$

*q-Fibonacci version: For $n \geq 0$,*

$$\sum_{k=0}^{n} \left(f_{k+1}(q)\right)^2 q^{\left(\binom{n+1}{2}-\binom{k+1}{2}\right)} = f_{n+1}(q)f_{n+2}(q) \tag{22}$$

*Proof.* Count pairs of weighted tilings of an $n$-board and an $(n+1)$-board and consider the location of their last common fault.

- **RHS:** The pair of weighted tilings is counted by $f_{n+1}(q)f_{n+2}(q)$.

- **LHS:** Assume the tilings have their last common fault at cell $k$. Then the tilings must have a series of staggered dominoes after cell $k$, these dominoes contribute $q^{\left(\binom{n+1}{2}-\binom{k+1}{2}\right)}$ and there remain $\left(f_{k+1}(q)\right)^2$ ways to tile the initial segments of the pair.

**Identity 2.9.** *Fibonacci version: For $n \geq 1$,*

$$f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n} - 1 \tag{23}$$

*q-Fibonacci version: For $n \geq 1$,*

$$\sum_{i=1}^{n} f_{2i}(q) q^{n^2-i^2} = f_{2n+1}(q) - q^{n^2} \tag{24}$$

*Proof.* Count the number of weighted tilings of a $2n$-board with at least one square.

- **RHS:** Clearly this is $f_{2n+1} - q^{n^2}$, all tilings except the tiling of all dominoes.

- **LHS:** Since there is at least one square, let $k$ be the position of the last square. Since only dominoes follow the first square, $k$ must be even. Let $k = 2i$. Then we have an arbitrary weighted tiling of length $2i - 1$ preceding the last square, contributing $f_{2i}(q)$ and the remaining tiles dominoes, contributing $q^{n^2-i^2}$, giving the desired result.

# 3 Remarks

Benjanmin et. al. and Vajda have produced extensive lists of Fibonacci, Lucas and Gibonacci identities [2], [3] and [6] inclusding dozens of identities involving Fibonacci numbers that can be generalized to give $q$-Fibonacci identities. The weighted tiling approach should be very useful in finding the $q$-analogues for Fibonacci identities not discussed here. In addition, similar combinatorial methods should be developed for $q$-analogues of Lucas numbers and Gibonacci numbers. Eric Lindgren, Mai Ahn Ngo and Stephen Segroves have done some of this work, finding several $q$-analogues of Fibonacci, Lucas and Gibonacci identities.

# References


[1] G.E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, Massachusetts, 1976.

[2] A. Benjamin, J. Quinn, *Proofs that Really Count*, The Mathematical Association of America, 2003.

[3] A. Benjamin, J. Quinn, F.E. Su, Phased Tilings and Generalized Fibonacci Identities,*The Fibonacci Quarterly* Vol. 38, No. 3 (2000) 282–288.

[4] E. Lindgren, S. Segroves, M.A. Ngo, private communication.

[5] I. Schur, Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche, *S.-B Preuss. Akad. Wiss. Phys.-Math. Kl.* (1917), 302–321, reprinted in I. Schur, *Gesmmelte Abhandungen*, Vol. 2, 117–136, Springer, Berlin, 1973.

[6] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section*, John Wiley and Sons, New York, 1989.



2000 *Mathematics Subject Classification.* 05A19, 11B39.

*Key words and phrases.* $q$-Fibonacci numbers, combinatorial proofs, weighted tilings.