Math 120A Name ____________________

December, 2003

Virtual Test III

Please show all work on these pages. **Answers without appropriate work will not receive credit.** You need not simplify your answers.

1. Find each of the following **derivatives**.

   (a) Find $f'(1)$ when $f(x) = 4x^3 - 3x^2 - 2x + 5$

   $$*f'(x) = 12x^2 - 6x - 2;\ f'(1) = 12(1)^2 - 6(1) - 2 = 4$$

   (b) Find $f'(x)$ when $f(x) = (4x^3 - 3x^2 - 2x + 5)(5x^3 + 3x^2 - 3)$

   $$*f'(x) = (12x^2 - 6x - 2)(5x^3 + 3x^2 - 3) + (4x^3 - 3x^2 - 2x + 5)(15x^2 + 6x)$$

   (c) Find $f'(x)$ when $f(x) = \frac{4x^3-3x^2-2x+5}{5x^3+3x^2-3}$

   $$*f'(x) = \frac{(12x^2-6x-2)(5x^3+3x^2-3)-(4x^3-3x^2-2x+5)(15x^2+6x)}{(5x^3+3x^2-3)^2}$$

   (d) Find $f'(x)$ when $f(x) = (4x^3 - 3x^2 - 2x + 5)^8$

   $$*f'(x) = 8(4x^3 - 3x^2 - 2x + 5)^7(12x^2 - 6x - 2)$$

   (e) Find $f'(x)$ when $f(x) = \cos((4x^3 - 3x^2 - 2x + 5)^4)$

   $$*\ f'(x) = -\sin((4x^3 - 3x^2 - 2x + 5)^4)\ (4(4x^3 - 3x^2 - 2x + 5)^3)\ (12x^2 - 6x - 2)$$

   (f) Find $f'(x)$ when $f(x) = \cos^4(4x^3 - 3x^2 - 2x + 5)$

   $$*f'(x) = 4(\cos^3(4x^3 - 3x^2 - 2x + 5))(-\sin(4x^3 - 3x^2 - 2x + 5))\ (12x^2 - 6x - 2)$$

   (g) Find $f'(x)$ when $f(x) = e^{\Pi} + x^e + \Pi^e$

   $$*\ f'(x) = e\ x^{e-1}$$

   (h) Find $f'(x)$ when $f(x) = \ln(4x^3 - 3x^2 - 2x + 5)$

   $$*f'(x) = \frac{12x^2-6x-2}{4x^3-3x^2-2x+5}$$

   (i) Find $f''(x)$ when $f(x) = \sin(5x)$

   $$*f'(x) = 5\cos(5x)$$

   $$*f''(x) = -25\sin(5x)$$

2. Find the following **antiderivatives**. Be sure to check your answers.

   (a) Find $F(x)$ when $F'(x) = 3x^5$

   $$*F(x) = \tfrac{1}{2}x^6 + C$$

   (b) Find $F(x)$ when $F'(x) = \frac{3}{x^5}$

   $$*F(x) = \frac{-3}{4x^4} + C$$

(c) Find $F(x)$ when $F'(x) = \frac{2xe^x - x^2e^x}{e^{2x}}$

$$*F(x) = \frac{x^2}{e^x} + C$$

(d) Find $F(x)$ when $F'(x) = e^x \cos\ x\ - e^x \sin\ x$

$$*F(x) = e^x \cos\ x + C$$

3. Two differential equations are each represented by one of the slopes fields below.
   (a) Which slope field (A or B) represents the differential equation $y' = y^2 - 1$? How do you know?

   * B is the correct slope field.
   *Note that $y' = 0$ when $y = 1$ or $y = -1$,
   * $y' < 0$ for y-values between -1 and 1
   *and positive elsewhere.

   (b) On the correct slope field, trace the solution of the initial value problem:

$$y' = y^2 - 1, \quad y(-1) = 0$$

*(p.41-42, iv and v)

4. While you were studying calculus in a 72-degree room, you got so enthralled by Newton's Law of Cooling that you forgot to drink the soda you had poured. Worse yet, you meant to take the temperature of the soda right after you took it out of the frig, since you are trying to determine if your refrigerator keeps liquids cold enough. The temperature of the soda was 60 degrees after 10 minutes and 62 degrees after 15 minutes. How cold was the soda when you first took it out of the frig? Note: A cup of cold liquid placed in a warm room warms according to Newton's law of cooling. If T_0 is the initial temperature, T_e is the temperature of the room, and $y(t)$ is the temperature at time t, Newton's Law of Cooling says that: $y' = k(y - T_e),\ y(0) = T_0$ which has the solution $y(t) = T_e + (T_0 - T_e)e^{kt}$

$$*T_e = 72,\ y(10) = 60,\ y(15) = 62$$
$$*y(10) = 72 + (T_0 - 72)e^{10k} = 60$$
$$\text{*and } (T_0 - 72)e^{10k} = 60 - 72 = -12$$
$$\text{*So } -12e^{-10k} = (T_0 - 72)$$
$$\text{*Similarly, } y(15) = 72 + (T_0 - 72)e^{15k} = 62$$

*which means $(T_0 - 72)e^{15k} = 62 - 72 = -10$

*So $-10e^{-15k} = (T_0 - 72)$

*Thus, $-10e^{-15k} = -12e^{-10k}$

*Or $\frac{-10}{-12} = \frac{e^{-10k}}{e^{-15k}}$

*$.83 = e^{5k}$ so $k = \frac{ln(.83)}{5}$

*Then $(T_0 - 72) = -10e^{-15\frac{ln(.83)}{5}}$

*So $(T_0 - 72) = -17.5$, and $T_0 = 54.5$

*Note: This is a harder problem than would be on the test.
*A more likely problem would give you the starting temperature
*and the temperature at 10 minutes and ask you
*to find the temperature at 15 minutes.

5. Evaluate the following limits:

(a) $\lim_{x\to\infty} \frac{5x^2 - 2 + x^4}{3x + 4x^4}$

$$*= \lim_{x\to\infty} \frac{\frac{5}{x^2} - \frac{2}{x^4} + 1}{\frac{3}{x^3} + 4} = \frac{1}{4}$$

*Or, by L'Hopital (which is correct but total overkill):

$$*= \lim_{x\to\infty} \frac{10x + 4x^3}{3 + 16x^3}$$

$$*= \lim_{x\to\infty} \frac{10 + 12x^2}{48x^2}$$

$$*= \lim_{x\to\infty} \frac{24x}{96x}$$

$$* = \lim_{x\to\infty} \frac{24}{96} = \frac{1}{4}$$

(b) $\lim_{x\to\infty} \frac{\sin(x)}{x}$

$$* = 0$$

(b-2) $\lim_{x\to 0} \frac{\sin(x)}{x}$

*By L'Hopital (necessary this time)

$$* \lim_{x\to\infty} \frac{\cos(x)}{1}$$

$$* = 1$$

(c) $\lim_{h\to 0} \frac{4(2+h)^3 - 4(2)^3}{h}$

* $f'(a)$ where $f(x) = 4x^3$ and $a = 2$

* $f'(x) = 12x^2$ so $f'(2) = 48$

6. A water storage tank with a square base is to be constructed to hold 12,000 cubic feet of water. If the metal top costs twice as much per square foot as the concrete sides and base, you want to find economical dimensions of the tank. Use derivative techniques to minimize the cost.

*Let x=side of base, h=height, c=cost per square foot of sides and base.

*Total= c(area of base and sides)+2c(area of top)

*$T = c(x^2 + 4xh) + 2c(x^2)$

*But $x^2h = 12000$, so $h = \frac{12000}{x^2}$

*So, $T(x) = c(x^2 + \frac{48000}{x} + 2x^2)$

*So, $T'(x) = c(2x - \frac{48000}{x^2} + 4x) = c(6x - \frac{48000}{x^2})$

*$T'(x) = 0$ when $6x - \frac{48000}{x^2} = 0$ or $x^3 = 8000$

*So, $x = 8000^{\frac{1}{3}} = 20$ feet and $h = \frac{12000}{20^2} = 30$

*Note, you can convince yourself that this is a minimum

*by showing that $T''(20)$ is positive, or simply by graphing T

7. A highway patrol officer stopped a motorist for speeding along a long strip of highway with a 55 mile per hour speed limit. "I never went faster than 55 mph since I left St. Olaf two hours ago," the motorist claimed. "Hah!" exclaimed the patrol officer. I know that St. Olaf is 112 miles away. I just learned about the Mean Value Theorem in calculus, and it proves that you did exceed the speed limit."

   (a) Let f(t) be the distance travelled in time t. Are the conditions of the Mean Value Theorem met for this function? How do you know?

   *Yes. The function is continuous and differentiable on [0,2].

   (b) Is the patrol officer right about the conclusions of the Mean Value Theorem? How do you know?

   *By the MVT, there is a c in [0,2] such that

   *$f'(c) = \frac{112-0}{2} = 56$

   *So somewhere in the interval, you were going 56 mph