

## BEYOND VULGAR MECHANICS: TECHNOLOGY IN TERTIARY MATHEMATICS EDUCATION

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### INTRODUCTION

What is mathematical computing for? Why do we bother with it in college mathematics?

The reason, I think, goes beyond the simple horsepower of doing more things more quickly. The real boon of mathematical computing, if we can achieve it, is to help our students (and ourselves) move beyond the routine mechanics to the deeper, more interesting, and more useful *ideas* of mathematics.

In a letter dated 25 May 1694, Isaac Newton wrote as follows (the spelling is Newton's):

A Vulgar Mechanick can practice what he has been taught or seen done, but if he is in an error he knows not how to find it out and correct it, and if you put him out of his road he is at a stand. Whereas he that is able to reason nimbly and judiciously about figure, force, and motion, is never at rest till he gets over every rub. . . . Experience is necessary, but yet there is the same difference between a mere practical Mechanick and a rational one, as between a mere practical Surveyor or Gauger and a good Geometer, or between an Empirick in Physick and a learned and rational Physitian.

The recipient was a certain James Hawer; Newton was discussing the ideal curriculum for a new school, arguing that a theoretical course in mechanics be required for practicing engineers.

I believe that mathematical computing, applied both nimbly and judiciously, can help us keep ourselves and our students never at rest; and it can help us make of our students (and of ourselves) not mere practical surveyors and gaugers, but learned and rational physicians, engineers, and perhaps even mathematicians.

### My History and “Credentials”

I should acknowledge from the outset that I'm something of an impostor here; I have no special knowledge of mathematics and computing in specifically Canadian education. But I hope and trust

that things north and south of the border aren't all that different. For that matter, I traveled south to get here to Brock.

In fact, I'm not completely unfamiliar with things Canadian, having grown up in another former British colony (India), enjoyed my honeymoon at the Empress Hotel in Victoria (in one of the less expensive rooms, suitable for graduate students), watched whales at the confluence of the Saguenay and St. Lawrence Rivers, vacationed at various times in Quebec, Ontario, Manitoba, British Columbia, and the Yukon. Indeed, students at the University of Washington, where I did my Ph.D, sometimes took me to *be* Canadian—something about the local accent, I suppose.

So much for my quasi-Canadian credentials. What about mathematics and computing?

I've been involved for a long time in calculus reform efforts in the States—some of them, in particular, having to do with more and different uses of computing, especially symbolic computing. Way back in 1986 (the year to which some people date calculus reform in the US), my college got a National Science Foundation grant to buy some early Sun workstations that ran SMP, a precursor of *Mathematica*; the machines cost around \$10,000—each. They did about what today's TI-89 and TI-92 calculators do, at less than 1/100th the cost in present dollars. (On the other hand, calculated on a *per pound* basis, those Suns cost about the same as today's calculators.)

The idea behind that early NSF grant was that, supplied with (what was then) modern computing, students in calculus and other college math courses would, more or less on their own volition, take the broader, more experiment-oriented approaches to these subjects that graphical, numerical, and symbolic computing can permit, and reap the concomitant benefits.

It didn't quite work out that way. True, the software was much less user-friendly and the hardware much more temperamental than now, and many students then (hard as it is to believe now) seemed cowed by those big, hulking machines, and quite a few couldn't even type. We could have solved those annoying but ultimately technical problems, or even just waited for them to solve themselves.

The harder problems were pedagogical and mathematical, not technical. We found it did little good, and perhaps even some harm to our credibility with students, to use the exciting new technology to teach the same old courses, with the same old agendas, from the same old textbooks. It felt like fitting an internal combustion engine to an oxcart. To use mathematical computing effectively, we found, students needed to be guided to think about mathematical ideas themselves—not just stereotyped manipulation problems—in entirely new ways, calling on graphical, numerical, and symbolic viewpoints as they were needed. “Old” calculus courses were simply too narrowly conceived and focused to support this broader perspective. These courses, at least as I encoun-

tered them, had a two-part agenda: first, to build symbol-manipulation facility with endless drill and practice on such things as finding symbolic derivatives and antiderivatives; and second, forcing occasional encounters with theorems, definitions, and other trappings of mathematical rigor. Both of these goals are probably Good Things, but they didn't really go well together. Taken too far, symbol-manipulation exercises (accept a symbolic string, transform it using some half-understood rules, and output a new symbolic string, called the derivative or antiderivative) can be more akin to word-processing than to mathematics.

This is not to deny that symbol-manipulation exercises can be challenging, and perhaps even valuable for some purposes, but they are a very long way from the proof-based, rigor-aimed approach that was supposed to be the other half of the agenda. (In my view, asking for fully rigorous proofs from beginning undergraduates is usually barking up the wrong tree anyway, but that's another story.) In any event, the practical effect was often to abandon the "idea" side of calculus entirely (usually with a litany of complaints about students' poor preparation, moral degradation, and general Philistinism) and to settle for (at best) a well-honed but narrow set of specialized symbol manipulation skills.

I think that's a bad bargain. We really can do better, and technology can help. The point, I think, is not simply, or even primarily, to race from symbol manipulation exercises to proofs, but to use technology to help students encounter in a deeper and more nuanced way the mathematical ideas that lie behind the symbols, and should properly come before proofs. To put it another way, we can use technology to help students "unpack" the ideas that are often so tightly compressed into mathematical symbols that students may miss them entirely. Symbolic computing is an important, and, on a broad educational scale, newly available tool—call it a can opener—that students can use to open and enjoy some of these tightly bound symbolic packages.

Though I'll often mention elementary calculus, I think the same principles apply to many other college mathematics courses.

Let me pause here to torch one straw man—the idea that calculators and computers must kill off ideas, reducing math to mindless button pushing. I find more logic in the claim that *no* calculators and computers can reduce math to mindless pencil pushing. There's nothing inherently wrong with pencil pushing—almost every mathematician I know actually enjoys it, just as musicians never stop noodling with their instruments—but let's not mistake pencil-pushing for real mathematics. The computers-dumb-down-mathematics schtick is simply wrong: almost all students and teachers I've talked with say that doing mathematics with computers is harder, not easier. Computer-based courses usually focus more on mathematical ideas, which, though certainly more interesting than routine manipulations, can also be much harder.

## FLEXIBLE THINKING: SOME EXAMPLES

Many examples have already been given at this conference of the sort of flexible, experiment-based thinking I think computing can help foster—hardly a surprising result when a Borwein is about. I'll add just a few more modest examples from elementary calculus.

**Example 1.** Plot and discuss the behaviour of functions of the form  $f(x) = x + a \sin x$ , for various values of the parameter  $a$  (e.g.,  $a = 1/2$ ,  $a = 1$ ,  $a = 2$ ). How do their graphs differ from each other? How do derivatives help explain the differences.

**Discussion.** Accurate graphics—readily available with any modern technology—are essential to the problem. Once these are available, most students (and most faculty) are surprised to find that  $f(x) = x + a \sin x$  is monotone increasing for both  $a = 1/2$  and  $a = 1$ ; only relatively large values of the parameter  $a$  successfully overcome the relentless monotonicity of the first summand. Taking symbolic derivatives (and second derivatives) immediately explains the results, and helps locate the critical points and points of inflection. Compare the richness of this activity with the sterility of simply writing down symbolic derivatives.

**Example 2.** Plot and discuss the behaviour of functions of the form  $f(x) = x^2 + a \sin x$ , for various values of the parameter  $a$  (e.g.,  $a = 1$ ,  $a = 2$ ,  $a = 10$ ). Use derivatives to explain the results.

**Discussion.** Again, accurate graphics are essential, and again there are surprises. (One is that the sinusoidal oscillation is almost invisible for small values of  $a$ .) A key insight comes from the *second* derivative, which is everywhere nonnegative unless  $a > 2$ .

**Example 3.** The series  $\sum_{k=1}^{\infty} 1/k^3$  converges, by the integral test, to some limit  $S$ . How large must  $n$  be to ensure that the  $n$ th partial sum  $S_n$  differs from  $S$  by less than 0.0001?

**Discussion.** Abstract convergence and divergence of series mystifies students; working hands on with real series and concrete numbers can help. Here, we can choose  $n$  so that the “tail”  $R_n = \sum_{k=n+1}^{\infty} \frac{1}{k^3}$  is less than 0.0001. Comparing the tail sum to an integral gives

$$R_n \leq \int_n^{\infty} \frac{1}{x^3} dx = \frac{1}{2n^2},$$

which is less than 0.0001 if  $n \geq 71$ . Hence  $S_{71} \approx 1.20196$  closely approximates the true limit.

**Example 4.** The functions  $f(x) = \frac{dx}{1+x^2}$  and  $g(x) = \frac{dx}{1-x^2}$  are typographically almost identical, but their antiderivatives,

$$\int f(x) dx = \arctan x \quad \text{and} \quad \int g(x) dx = \frac{1}{2} \ln \frac{x+1}{x-1}$$

are very different. What's going on?

**Discussion.** Plotting the integrands is a first step; it reveals the vertical asymptotes at  $x = \pm 1$ , and suggests the presence of logarithmic terms in an antiderivative. Expansion in partial fractions—an easy task for (say) *Maple*—shows exactly where the logarithmic terms come from.

### Where Are We Now?

Mathematical computing, including graphical and numerical capabilities, has been available for use in teaching calculus and other undergraduate courses for around 10 years. What has changed? (Again, calculus is my prime example, but it's probably a harbinger of things to come elsewhere in the tertiary curriculum.)

**Graphics.** Perhaps the most noticeable change so far from the use of calculators is students' willingness—indeed, eagerness—to use graphical evidence. In the past, students seldom used graphs other than those explicitly required by a problem. Now students are quite likely to draw a graph voluntarily—of a function to be maximized, for example—to make sure an answer is reasonable. Couldn't they have done this without a calculator? In theory, yes, but in practice they seldom did so, either for lack of experience with this kind of thinking or because it simply didn't seem worth the trouble. (It was a lot more trouble to draw graphs back then.) Thus the presence of calculators has concretely changed the ways many students do and think about mathematics. Books have changed, too: problems of the form “graph this function, then quit” are gone or much reduced, even in relatively traditional books. In their place are much more substantial and mathematical problems, of the form “here's a graph, do something with it, or tell me something about it.”

Ironically, we now see more problems that are truly non-technology problems, in the sense that technology doesn't render them pointless. Problems with graphically-presented functions (for instance: given the graph of a function  $f'$ , discuss properties of the original function  $f$ ) are technology-proof in the sense that pushing buttons doesn't help. However, these problems often depend on

insights that are crucially fostered and supported by technology. Looking through some older calculus textbooks, such as *Calculus and Analytic Geometry*, G.B. Thomas, 4th edition, (I bought it for \$6.95 in 1968) reveal that a large fraction of the problems in these books are now, by any reasonable measure, nothing but “calculator problems,” since symbolic computing renders so many of them pointless or silly. (There are also a lot of theoretical problems in the old textbooks, but I don’t remember either assigning or doing many of them.) Thus there’s a bracing sense in which the technology helps take the whole calculus enterprise back to its old-fashioned roots. In this sense, some of the apparently new-fangled problems in today’s textbooks are really classical, even timeless: they’re about the ideas of calculus, not the calculations.

Differential equations offer another example of technological leverage. The importance of differential equations in first-year calculus and other early courses is increasing, thanks partly to the urging of our colleagues in other disciplines, who use DEs every day, and partly, I’d like to think, to the simple truth that DEs are far and away the most important “real-world” application of calculus ideas. (If calculus were good for nothing more than the usual “applications,” such as finding volumes of solids of revolution, the subject would have died out of curricula centuries ago, and deservedly so.) As DEs become more important, so does the value of technology, especially for conveying the qualitative behavior of solutions. The subject of DEs is notoriously difficult to convey effectively without technology, because the highly condensed form of the traditional symbolic representation makes the ideas difficult for a newcomer to unpack. With technology available to illustrate such things as slope fields and phase planes, and even to find symbolic solutions in some cases, the rudimentary ideas of DEs become enlightening, even straightforward.

**Numerics.** The examples above suggest that it is graphical technology that has, so far, had the greatest effect. By contrast, numerical technology and numerical methods seem less readily appealing to students. Although numeric manipulations have been available on cheap handheld machines—scientific calculators—for a long time, they have had little effect on mainstream calculus courses. While numerical methods and viewpoints are certainly important in their own right, they haven’t yet become central enough in calculus and other beginning courses, or perhaps interesting enough to students, to change significantly the way the subjects are approached. For example, even students who have learned calculus using calculators with graphics and numerical integration capability will often thrash about for antiderivatives when asked for a definite integral of  $\exp(-x^2)$ . Indeed, numerical methods may even strike students as cheating, since they don’t produce exact answers. (Graphs don’t either, of course, but students seem willing to forgive—or overlook—that problem.)

**Symbolics.** Although mathematical computing has already begun to reshape college mathematics courses, the largest and most important changes are yet to come. The important new feature of mathematical computing, only recently available cheaply and at large scale—is computer symbolic manipulation. Computer symbolics will probably transform college mathematics courses at least as much as graphical computing has done, and, from a mathematical point of view, probably in more fundamental ways. Symbolic computing impinges on an even larger part of the traditional agenda of college mathematics courses—formal manipulation of elementary functions—than does graphical computing, and so raises harder and larger questions of content, method, and pedagogy.

Let's make no mistake: computer symbolic manipulations in students' hands could revolutionize our courses, whether we like it or not. Symbolic technology confronts us with the same alternatives as graphing technology has already done at lower levels: we can take account of technology and fundamentally rethink our courses, or we can ignore it and marginalize mathematics courses in too many students' eyes.

**Attitudes.** Although symbolic manipulators are not (quite) yet in common use by students, there has already been a noticeable change in students' attitudes toward the value of symbolic manipulation. Deborah Hughes-Hallett describes a problem that has long been set for Harvard freshmen in Calculus II (many of whom have symbol-manipulating calculators at hand): finding the arc length of a circle by integration. This involves differentiating a square root, using the chain rule, squaring, simplifying, and antidifferentiating the resulting algebraic fraction. In olden days, this manipulation caused nary a murmur; if students made mistakes, they blamed themselves. Now the problem causes complaints: the manipulations make students groan and they're impatient to get on to the answer.

What does this impatience tell us? Quite possibly, the very existence of the symbolic technology is already cultivating in students a lesser respect for manipulations that can be readily be done by machine. There seems to be a growing, if still inchoate, perception that courses that teach only manipulations and algorithm performance are not worth anyone's time.

We should not, of course, leap too quickly to the conclusion that hand and mental symbolic manipulations are, across the board, no longer worth teaching or knowing. But if we are to succeed in teaching an appropriate balance of manipulations and concepts, we will have to rethink our courses thoroughly, making a convincing case that what manipulations we do teach are necessary—even in a technology-rich world.

Faculty attitudes need some changing, too. Complaining about students' lack of symbolic fa-

cility and their too-hasty resort to technology is a favorite coffee-room sport, and not likely to go out of fashion soon. But there's another side to the story. We should take advantage of the fact that our students enjoy and are generally good at using technology. We should be able to harness technology, and students' fascination with it, to draw students into mathematics. Using it creatively in our teaching, we may find we teach much more and more substantial mathematics by drawing on our students' reservoirs of technical skill, of which they are deservedly proud.

### **What's Ahead?**

It seems clear that symbolic computing is coming to the masses, and coming soon. What the large effects will be remains to be seen. Here are some guesses and open questions.

1. The fear is sometimes expressed that technology renders mathematics a mindless mechanical exercise.

Exactly the opposite effect is much more likely. With machines to do more of the calculation, calculus and other introductory courses can become more, not less, focused on ideas and concepts. These courses may well become "harder" in the process. What are the implications of that?

2. Symbolic computing raises serious questions about the proper role and placement of algebra in the mathematics curriculum. The traditional role of calculus as remedial algebra is probably over. In itself, that's probably a good thing for calculus, but what is the legitimate purpose of algebra? What algebra? When will students learn what they need?
3. What are the differences between "necessary symbolic manipulation" and "mindless symbolic manipulation"? How do we recognize each "in the field"?
4. What are the proper roles of mental, paper-and-pencil, and machine-based calculations, especially when students who begin tertiary mathematics are accustomed to performing algebraic manipulations using computers or calculators?
5. When symbolic manipulators become common, what parts of the traditional symbolic agenda of such courses as elementary calculus and linear algebra will still be worth doing, and why? Some of the answers we settle on might surprise us. Partial fractions, for instance, is often symbolically difficult for students, but it is also a beautiful divide-and-conquer technique. Computer algebra can certainly make the method more accessible; arguably, the method becomes more important, too. Could Cramer's rule come back to life?

6. How well will mathematics majors who learn calculus and other beginning course with technology be prepared for real analysis and other upper-level, proof-based math courses? Will the greater “conceptual” focus of these earlier courses foster better theoretical thinking in later courses, or will it spawn sloppiness?
7. What will mathematical exercises, exam questions, and competitions look like 10 years from now?

Hard questions indeed—but mathematics is about hard questions.

I started by quoting Newton; I’ll end by quoting an annoying TV commercial:

Opportunities abound. Let’s get to work.

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