Humke’s Take-home Exam 1
April 26, 2002

Instructions: You must show and explain all of your work. Err on the side of providing too much explanation rather than too little. Your proofs should be clear and well-written. Please include your final draft only; don’t include scratch or preliminary work with your final draft.

You may consult sections 0.1 through 3.3 of the textbook, your calculus and linear algebra textbooks, and your own course notes, but you may not look at other books nor anyone else’s notes. You should not talk to anyone other than myself about this exam. You may ask clarifying questions either in person or via e-mail.

You may reference theorems that we have discussed in class or you have proved in homework, but all other results must be proved. One exception to this is if I ask you to prove something we happen to have already proved, then you must prove it again from scratch.

Good Luck! Start this before Tuesday night! Staple this sheet to your work and turn everything in by 3pm on Friday, May 3.

Pledge: I pledge my honor as a gentle person that I have neither given nor received information concerning this examination and that I have seen no dishonest work.

Signed: 

Name: 

☐ I have seen the pledge and choose not to sign.
1. *Category I.* Do two of the following problems.

(a) Let \( G = \{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c, d \in \mathbb{R}, \text{ and } ac \neq 0 \} \). Show that \( G \) is a group under matrix multiplication, but not under addition.

(b) Let \( \mathbb{R} \) be the group of all real numbers under addition, and let \( \mathbb{R}^* \) be the group of *positive* real numbers under multiplication. Define \( \phi: \mathbb{R}^* \to \mathbb{R} \) by \( \phi(x) = \log(x) \) (where \( \log \) is the log function base \( e \)). Prove that \( \phi \) is an isomorphism.

(c) Let \( G \) be a group and let \( a \) be a fixed element in \( G \). Define \( H = \{ x \in G | ax = xa \} \). Prove that \( H \) is a subgroup of \( G \).

2. *Category II.* Do two of the following problems.

(a) Let \( \phi: G \to G' \) be a group homomorphism from the finite group \( G \) onto the group \( G' \). Suppose \( |G'| = p \) where \( p \) is a prime. Prove that \( |G| \) is a multiple of \( p \).

(b) \( U_{90} \) has a very limited number of subgroups and due to the *Fundamental Theorem of Finitely Generated Abelian Groups*, each subgroup of \( U_{90} \) is isomorphic to a product of cyclic groups, \( \mathbb{Z}_n \times \mathbb{Z}_m \times \ldots \). Which products of cyclic groups are isomorphic to subgroups of \( U_{90} \)? Make certain (prove) your list is all inclusive.

(c) Suppose \( N \triangleleft G \) and \( \forall a, b \in G, aba^{-1}b^{-1} \in N \). Prove \( \frac{G}{N} \) is ableian.

3. *Category III.* Do two of the following problems.

(a) Prove there are no nonabelian groups of order 9.\(^1\)

(b) If \( H < G \) and \( N \triangleleft G \), prove that
   i. \( H \cap N \triangleleft H \)
   ii. \( HN = \{hn : h \in H \text{ and } n \in N \} \). Prove \( HN < G \).

(c) Let \( G = \mathbb{Z}_{12} \times \mathbb{Z}_{15} \) and \( H = \langle (4,5) \rangle \). Then \( \frac{G}{H} \) is isomorphic to a product of cyclic groups. Which product of cyclic groups is it?

\(^1\)Extra Credit Does your result generalize to groups of order \( p^2 \) where \( p \) is a prime?