1. Determine whether the following subsets are subgroups of the given group.

(a) The set of even integers as a subset of the group \((\mathbb{Z}, +)\).
(b) The set of odd integers as a subset of the group \((\mathbb{Z}, +)\).
(c) \(\{x \in \mathbb{R} \mid |x| = 1\}\) as a subset of the group \((\mathbb{R} - \{0\}, \cdot)\).
(d) What must be true about \(b\) if the set of points on the line \(y = mx + b\) is to be a subgroup of the plane \((\mathbb{R}^2, +)\)?

2. Let \(G\) be a group and recall that \(Z(G) = \{x \in G \mid xg = gx, \forall g \in G\}\) is the center of \(G\). Prove that \(Z(G)\) is a subgroup of \(G\) by following the steps below.

- **Closure:** Let \(a, b \in Z(G)\). We must show \(ab \in Z(G)\).
- We show this by showing \(abg \in Z(G)\) for all \(g \in G\).
- Now
  \[
  (ab)g = a(bg) \quad \text{by associativity}
  = (ag)b \quad \text{since } b \in Z(G)
  = gb = (ab)g \quad \text{by }
  = g(ab) \quad \text{by }
  
- **Inverse:** Let \(a \in Z(G)\). We must show \(a^{-1} \in Z(G)\) by showing \(a^{-1}g = ga^{-1}\) for all \(g \in G\).
- Since \(a \in Z(G)\), we know \(ag = ga\) for all \(g \in G\). Multiplying both sides of this equation on the left by \(a^{-1}\) gives \(a^{-1}ga = a^{-1}\).
- Now multiply this new equation by \(a\) and we get \(a^{-1}g = a^{-1}g\) for all \(g \in G\). Thus, we see that \(a^{-1} \in Z(G)\).

3. Consider the group \(S_3\) with Cayley table given below.

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<tr>
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<th>(e)</th>
<th>(\sigma)</th>
<th>(\sigma^2)</th>
<th>(\tau)</th>
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</tbody>
</table>

(a) Find all the elements in the 6 cyclic subgroups of \(S_3\).

\(\langle e \rangle = \langle \sigma \rangle = \langle \sigma^2 \rangle = \langle 1 \rangle\)

\(\langle \tau \rangle = \langle \sigma \tau \rangle = \langle \sigma^2 \rangle = \langle 1 \rangle\)
(b) Is $S_3$ a cyclic group? (i.e., is $S_3$ generated by any of its elements?)
(c) Is $\sigma \in Z(S_3)$? Find all elements in $Z(S_3)$.
(d) Identify all elements in the set $\{x \in S_3 \mid x^2 = e\}$. Does the set form a subgroup of $S_3$?
(e) Identify all elements in the set $\{x^2 \mid x \in S_3\}$. Does the set form a subgroup of $S_3$?
(f) The centralizer of an element $x$ in a group $G$ is the set $C_G(x) = \{g \in G \mid xg = gx\}$, which is the set of elements which commute with just $x$. (You will prove that $C_G(x)$ is a subgroup of $G$ in your homework.) Find $C_{S_3}(x)$ for all $x \in S_3$.

4. Let $G$ be a group. Under what conditions is $H = \{x^2 \mid x \in G\}$ a subgroup of $G$?

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**Math 252–Homework 8**

Due Friday, March 1

- Read §1.4 and do §1.4/8,11,12,15,16,20,22,23,27,28,36 (For 8, 11, 12, recall that $\det(AB) = \det(A) \cdot \det(B)$.)
- On a separate page, do §1.4/43,48,51 (For 43, note that an element of the set is “something from $H$ times something from $K$” in that order.)
- **Extra Credit**: Any of §1.4/41,42,49,57