Sequences and Limits of Sequences

Math 301

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Definition 1. Basic Sequence Definitions.

1. A sequence is a function whose domain is \( \mathbb{N} \).
2. A real valued sequence is a function \( x : \mathbb{N} \to \mathbb{R} \);
3. A complex valued sequence is a function \( x : \mathbb{N} \to \mathbb{C} \).
4. The “\( n \)th term of \( x \) is \( x(n) \) and is typically denoted with a subscript rather than parentheses, i.e. the first term is \( x_1 \) rather than \( x(1) \).

We’re interested in limits of sequences and the idea of such a limit seems simple enough: something like

\[ x_n \text{ gets closer and closer to } L \text{ as } n \text{ gets increasingly large.} \]

However, such a definition can be misleading and downright confusing. For example, the sequence \( \{1 - \frac{1}{n}\} = \{1, \frac{1}{2}, \frac{2}{3}, \ldots\} \) is getting “closer and closer” to 10 as \( n \) gets increasingly large, but the limit is not 10 at all. Rather than “closer and closer” to what we really need is something like “as close as possible to.” But this too is misleading since no number can be as “close as possible” to another number. (If \( a < b \), then there are both rationals and irrationals between \( a \) and \( b \).)

So the actual definition of the limit concept is a bit more obtuse than one might expect at first glance; here it is.

Definition 2. A sequence, \( \{x_n\} \) converges to a limit \( L \) if

for every \( \epsilon > 0 \) there is an \( N \in \mathbb{N} \) such that when \( n \geq N \), \( |x_n - L| < \epsilon \).

This is denoted by

\[ \lim_{n \to \infty} x_n = L \text{ or equivalently } \{x_n\} \to L. \]

Problem 1. Here are some problems to check yourself on.

1. Prove \( \{\frac{1}{n^2}\} \to 0 \).
2. Prove \( \{\frac{1}{n^2 + 1}\} \to 0 \).
3. Prove \( \{\frac{1}{3n^2 + 2n - 6}\} \to 0 \).
4. Prove \( \{\frac{2n - 5}{3n^2 - n - 3}\} \to 0 \).
5. Prove \(\frac{2n^2-n-5}{4n^2-n-3} \to \frac{1}{2}\).

6. Prove \(\frac{3n^5-8n^2+n-9}{n^7+n^4-3n+4} \to \)???

Here are three or four illuminating facts concerning limits of sequences.

**Theorem 1.** *Monotone Convergence Theorem for Sequences.*

- Every bounded increasing\(^1\) sequence converges.
- Every bounded decreasing sequence converges.

**Proof.**

Case 1. Suppose that \(\{x_n\}\) is increasing and bounded above by \(B > 0\). Then the range of the sequence \(\{x_n\}\) is contained in \([0, B]\) and hence the range has a least upper bound, say \(L\).

**Claim** The sequence \(\{x_n\}\) \(\to L\).

**Proof.** Let \(\epsilon > 0\) be given. Since \(L - \epsilon < L\) and \(L\) is the least upper bound for the range of \(\{x_n\}\), \(L - \epsilon\) is NOT upper bound for \(\{x_n\}\). Hence, \(\exists N \in \mathbb{N}\) such that \(L - \epsilon < x_N\). Ah Ha!! Now let \(n \geq N\). Since \(\{x_n\}\) is increasing, \(x_N < x_n\) and as \(L\) is an upperbound for \(\{x_n\}\), \(x_n \leq L\). Putting all these together we find that for \(n \geq N\),

\[
L - \epsilon < x_N < x_n \leq L < L + \epsilon.
\]

Thus, for \(n \geq N\), \(|x_n - L| < \epsilon\) as required. \(\square\)

Case 2. (Prove this case for practice.)

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\(^{1}\)“increasing” means \(x_n \leq x_{n+1}\ \forall n \in \mathbb{N}\).
Theorem 2. If a sequence does not have an increasing subsequence\(^2\) then it has a decreasing subsequence.

**Proof.** Let \( S \) be the set of “local max’s” for the sequence, \( \{x_n\} \); that is:
\[
S = \{x_n : x_n \geq x_m \text{ whenever } m > n\}.
\]
If \( S \) is infinite, then \( S \) is a decreasing subsequence. The other possibility is for \( S \) to be finite and in this case, there is an \( N \in \mathbb{N} \) such that for \( n \geq N, x_n \not\in S \). Under these circumstances, we can find an increasing subsequence of \( \{x_n\} \) using Mathematical Induction as follows.

**Step 1.** Let \( y_1 = x_{N+1} \). (The main point here is that \( y_1 \not\in S \).)

**Step 2a.** Suppose \( y_k \) has been selected from \( \{x_n : n \geq N + 1\} \).

**Step 2b.** Since \( y_k \in \{x_n : n \geq N + 1\} \), there is an index \( m \geq N + 1 \) such that \( y_k = x_m \). Since \( m \geq N + 1 \), \( x_m \not\in S \). Hence, there is an \( m^* > m \) so that \( x_m < x_{m^*} \). Let \( y_{k+1} = x_{m^*} \). Then \( y_k < y_{k+1} \).

This finishes the induction and it follows that in this case, \( \{y_k\} \) is an increasing subsequence of the original \( \{x_n\} \). \( \square \)

**Theorem 3.** *(Balzano-Weierstrass)* Every bounded sequence has a convergent subsequence.

**Proof.** Hey, this is the famous *BW Theorem*.\(^3\) Can you use the stuff on these pages to prove it? Hint – The proof is just three sentences!

**Problem 2.** In each of the following problems use Mathematical Induction to prove that the recursively defined sequence is decreasing\(^4\). Also show the sequence is bounded. Conclude that the sequence converges and use algebra to find the limit.

1. \( x_{n+1} = \frac{1}{4}(x_n + \frac{x_n}{x_n}) \) and \( x_1 = 2 \).

2. \( x_{n+1} = \frac{1}{2}(2x_n + \frac{1}{x_n}) \) and \( x_1 = 2 \).

\(^2\)a subsequence is a selection of the terms taken in order

\(^3\)Also sometimes irreverently called the Banana–Weanyroast Theorem.

\(^4\)this is cool – a bit on the sneaky side!