Goals and content

What are math/science/engineering-oriented calculus courses for? What should calculus students know? What should they do? What applications should they see, and why? How will we help students achieve our goals for them? How are these questions related to each other, and to the present and future state of mathematical computing? What happens next?

The goals and content of calculus courses are, of course, tightly intertwined. To a large extent, the content of a course represents the instructor’s practical strategy for achieving its larger goals. So I’ll discuss goals and content mainly together, mentioning either or both as they arise.

Some legitimate goals of calculus, however, have little to do with specific content choices. For instance, mathematics courses in general, and calculus courses in particular, are sometimes said to build mental discipline, introduce the mathematical method, and (more grandiosely) teach students “how to think.” This sort of talk sounds a little passé nowadays; we’re probably more comfortable talking about interesting applications of calculus and about how advanced mathematics builds fundamentally on calculus. But the old “mental training” agenda is still perfectly valid, in my opinion. It may gain even more weight in the future, as both applications and the field of mathematics change, perhaps in ways we haven’t yet dreamed of. Mathematical computing is the obvious change engine right now, but it’s far from obvious that that will continue. What’s ahead, for example, if genetics and neurobiology continue to grow at their present rates? Today’s applications may soon seem quaint, but mathematical ways of thinking will endure.

Calculus for whom?

In prognosticating about what calculus courses should do and be, we should be clear on our audience(s). Are we addressing mathematics majors? Engineering majors? Students who’ve already seen some calculus in high school? Well-designed calculus courses needs to take due account of the different needs of these different groups.

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1I had better be careful in talking about strategy and tactics here at the Military Academy . . . .
One possibility is to choose a specific clientele, focusing narrowly on its special needs and interests. Another is to address a diverse clientele (as is usual in liberal arts colleges), stressing broad ideas and principles that cut across areas of application. Which of these situations obtains governs many important design choices: whether mathematical models are “built” or only “used”; the level and role of proofs and mathematical rigor; the importance of limits; the choice of applications; the balance of numerical and symbolic approaches; the curricular prominence of DE’s; the importance of developing symbolic facility (by hand, by head, or by machine); etc.

My hunch is that, with some important and useful exceptions (including the very sort of integrated mathematics/science courses being discussed right here at this conference), most beginning calculus courses will continue to serve relatively broad student audiences, and will have relatively broad educational goals. Calculus can be many things: a general tool for mathematical applications; the end of a long and tedious road through pre-calculus; an entry point to the mathematics major; a “general education” introduction to mathematical culture and thought. For many students, the course serves several of these functions.

I would hazard the companion guess, therefore, that mainstream calculus courses of the future will hew more, not less, to basic mathematical ideas and concepts (which serve a wide and growing range of disciplinary and pedagogical goals) than to specific areas of application. If I’m right, a challenge for the future will be to find not only new and attractive applications of calculus but also, and just as important, ways of making the good, old, useful, basic ideas of calculus clearer and more applicable.

The two-fold way

In thinking about calculus content and goals, it may be helpful to recall two quite different basic approaches to the subject. One strategy—let’s call it the math way—is to emphasize limits, which are indisputably the fundamental mathematical objects on which a rigorous development is based. Another road to the calculus—let’s call it the science way—is to take rates of change, not limits, as the course’s main “primitives,” leaving to later work (and, therefore, mainly to mathematics majors) such subtler and less intuitive issues as defining rates rigorously (as limits!), proving existence, and the like.

To illustrate the difference, consider the tangent line problem. The science way—zoom in on a nice graph until it looks straight—takes existence questions for granted (or defers them to later courses), and worries mainly about meaning and interpretation. The math way forces the hard questions about definition and existence, and answers them using limits.

The science way has clearly been in the lead for the last 15 years, since what we call calculus reform came along. It may be less obvious, however, that the science way has been on the rise for over a century, with calculus authors (including such true luminaries as Augustus
de Morgan) arguing for a physically intuitive, rates-based approach, rather than one based on a rigorous theory of limits. The same rates-based approach characterizes Elias Loomis’s *Elements of Analytical Geometry and of the Differential and Integral Calculus*, one of the first American textbook treatments, published in 1852.

I like the rates-based approach to beginning calculus, and I see no sign of it changing. (This may sound like heresy, coming from a mathematician, and an analyst to boot. But I think that mathematical analysis, like certain other rare and keen pleasures of life, is best enjoyed at an appropriately advanced “age.”) One of our tasks for the future may be to find problems and applications that both build on and advance a rates-based understanding of the subject.

**Symbolic facility and symbol sense**

What balance of skills and concepts is necessary to use calculus effectively in science and mathematics? I find the question highly non-trivial—especially in the presence of symbol-manipulating technology tools. One answer—that once a machine can do something, humans shouldn’t—I find neither pedagogically nor practically convincing. (In a vaguely related vein, I’ve heard that the U.S. Naval Academy still requires beginners to master sail-driven craft—a nice idea.) The key, I think, is to link skills with concepts, choosing the former to strengthen the latter.

For instance, I see little value in spending one’s limited course budget of time and energy on complicated partial fractions problems (e.g., ones that involve powers of irreducible quadratics)—that may as well be left to Maple or Mathematica. But the idea of partial fractions and some simple examples are probably as important as ever. There’s little typographical difference between the functions

\[ f(x) = \frac{1}{1 - x^2} \quad \text{and} \quad g(x) = \frac{1}{1 + x^2}, \]

but, in every other way, the functions (and the phenomena they might model) are completely different, as are their antiderivatives: the functions

\[ \int f(x) \, dx = \frac{\ln|1 + x| + \ln|1 - x|}{2} \quad \text{and} \quad \int g(x) \, dx = \arctan x. \]

The partial fraction decomposition goes a long way toward explaining what’s happening. (Looking at graphs is helpful, too, of course, but graphs alone don’t explain why logs and arctangent, rather than other functions with approximately the same shapes, are involved.)

For some students it’s enough, at least on the first pass, to learn mainly about calculus, concentrating on its main objects and ideas more than on its manipulations. Most mathematics and science students, on the other hand, have a more ambitious “facility agenda”—they
need, sooner or later, to develop enough speed and confidence to do calculus efficiently, and use it as a sharp tool to solve problems in other areas.

This sort of mathematical facility—what my 9th grader and I secretly call “kick-ass” math—has always required some skill and ease in throwing symbols around on paper or in one’s head. I think it always will—even when every student has symbol-manipulating technology at her elbow. (Or perhaps on her wrist ... a small company in my home town is busy developing an Internet-ready Dick Tracy watch.)

What symbolic facility means may well change. In the past, it meant things like expanding complicated rational functions in partial fractions, factoring polynomials, or grinding out tricky symbolic antiderivatives by hand. In the future, symbolic facility may mean other things, such as anticipating the form of an answer (e.g., the logarithmic and arctangential ingredients in rational function antiderivatives, the nested form of a composite derivative, etc.), or noting its absence with concern.

To help students build this facility, or symbol sense, we will probably continue to assign symbolic exercises, but perhaps in new forms, designed to foster pattern recognition and to point out structure as much as to crank out symbolic results. For example, I’d like to see students understand and visualize better the effect of parameters on function families like \( f_a(x) = ax + \sin(x) \) and their derivatives. I’d gladly trade some of this for, say, trigonometric substitutions.

We might also aim to connect symbolic representations and operations more directly and concretely to graphical representations—a goal made more attainable by technology that handles numbers, pictures, and symbols.

Content

What specific content changes and emphases might support a “modern” beginning calculus course for science and mathematics? Here are some guesses:

**Differential equations.** Although differential equations and initial value problems are unquestionably the most useful calculus tools for the clientele at hand, they’ve played oddly minor roles in calculus courses so far. The basic ideas behind DE’s, IVP’s and their solutions are entirely accessible to calculus students, even if sophisticated symbolic solution techniques are not. With graphical and numerical tools available, DE’s will probably achieve the more prominent place they deserve.

**Infinite series.** Modern beginning courses may well de-emphasize infinite series, at least as regards abstract questions of convergence and divergence. At the same time, more attention may be due to (related) issues of approximation. For example, whether
a Taylor series approximation actually converges to the right place may be less of interest than how closely Taylor polynomials approximate the target function. This is by no means to deny that series, convergence, and divergence are lovely “proto-analysis” topics for mathematics majors. But for most students these topics “belong” at a later developmental time.

**Multivariate topics.** We may soon pay more and earlier attention to multivariate topics of various sorts. (I think Don Small will agree with me.) These could include multiple integrals, partial derivatives, parametric equations, and the like. These are probably more important to most non-math majors, and more likely to be encountered later, than (say) abstract convergence and divergence, which they might well displace in one-year courses.

**Discrete and dynamical systems.** Difference equations are natural counterparts to DE’s. With even modest computing resources at hand, students can investigate discrete dynamical systems, use them in modeling, and perhaps begin to see how discrete and continuous viewpoints complement each other.

**Setting high standards**

Among the bum raps sometimes adduced against calculus reform, the low standards rap may be the bummest. Ask almost any student, almost anywhere, who has experience in both standard and reformed courses, which is “harder.”

Still, the question of high standards is fair. What are they? How will we achieve them?

High standards are, at one level, like motherhood and apple pie—who could oppose them? But what does the phrase mean? What should it mean? In times past, high standards in calculus (e.g., in honors calculus) tended to mean traditional mathematical rigor: precise definitions, careful proofs, development based carefully on limits, and mainly mathematical applications. High-standards calculus was, in essence, the fast track to real analysis.

But there’s a broader view of high standards, toward which I think we’re (appropriately) tending, driven partly by computing possibilities and partly by different uses our students will make of calculus. High-standards calculus courses of the near future could be characterized by such things as more and deeper modeling problems (where students build, not just use, calculus-based models); more writing and verbal presentation; more open-ended, investigative activities; more challenging applications. (The traditional proto-analysis course will continue to exist, but as only one option.)

High standards are not only for the gifted—they should apply in all courses. The “hard” stuff in standard courses has traditionally been symbol manipulation—trigonometric substitution, complicated partial fractions, nested composite derivatives, some timid stabs at
epsilon and delta. But those aren’t the only possibilities, or the best ones. We can challenge all students, not just in honors courses, with appropriate versions of the features mentioned above. All students need to understand what they’re doing and why—not just how to calculate.