1 Introduction.

Let $f : X \to X$ be a continuous map on the compact metric space $X$. We let $C(X, X)$ be the space of all continuous functions from $X$ into $X$.

Of considerable interest and importance in dynamical systems are minimal sets. A set is minimal for $f$ if it is a minimal, nonempty, closed invariant subset of $X$ with respect to $f$.

Our interest lies in adding machines, a particular class of minimal sets also referred to as solenoids or odometers. These are all Cantor sets topologically. In a minimal set each point is strongly recurrent. This is a characterization which is a classical result of Birkhoff ([2], [3]). Blokh and Keesling characterize adding machines as those infinite minimal sets in which each point is regularly recurrent.

Agronsky, Bruckner and Laczkovich show in [1] that given a generic continuous self-map $f$ of the unit interval $I$ there is a residual set of points $x$ in $I$ for which the $\omega$-limit set $\omega(x, f)$ is a Cantor set. Using a much different approach, Lehning extends these results to continuous self-maps of any compact $n$-dimensional manifold [7]. In [8] Steele goes a step further by showing that, on the interval, a generic ordered pair $(x, f)$ gives rise to an $\omega$-limit set generated by an adding machine. Our work can be viewed as an extension of
[1], [7] and [8] as we analyze the structure of the adding machines generated by a generic continuous self-map of $M$ where $M$ is a $n$-manifold or the Cantor space. We analyze the structure of the adding machines generated by a generic continuous self-map of $M$ where $M$ is a $n$-manifold or the Cantor space. In particular we answer the following:

**Query:** What can we say about $\omega(x,f)$ for the typical $(x,f) \in X \times C(X,X)$, and what can we say about $L = \{\omega(x,f) : x \in X\}$ for the typical $f \in C(X,X)$?

2 Adding Machines, Solenoids, Odometers.

Much of the terminology is borrowed from [4]. If $\alpha \in (N \setminus \{1\})^N$, set

$$\Delta_\alpha = \prod_{i=1}^{\infty} Z_{\alpha(i)},$$

where $Z_k = \{0, \ldots, k - 1\}$.

Instead of the usual coordinate-wise addition, we add two elements of $\Delta_\alpha$ with “carry over” to the right. More precisely, $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ in $\Delta_\alpha \Rightarrow$

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (z_1, z_2, \ldots),$$

where

$$z_1 = (x_1 + y_1) mod(\alpha(1))$$

and, in general,

$$z_i = x_i + y_i + \epsilon_{i-1} mod(\alpha(i))$$

where

$$\epsilon_{i-1} = \begin{cases} 
0 & \text{if } x_{i-1} + y_{i-1} + \epsilon_{i-2} < \alpha(i-1) \\
1 & \text{otherwise.}
\end{cases}$$

If we let $f_\alpha$ be the “+1” map, that is

$$f_\alpha(x_1, x_2, \ldots) = (x_1, x_2, \ldots) + (1, 0, 0, \ldots),$$

then $(\Delta_\alpha, f_\alpha)$ is a dynamical system known in various contexts as a $\alpha$-adic solenoid, adding machine or odometer. These are all Cantor sets.
Definition 2.1 ([4]). Let $\alpha \in (N \setminus \{1\})^N$. We define a function $M_\alpha$ from the set of primes into $N \cup \{\infty\}$ as follows. For each prime $p$, let

$$M_\alpha(p) = \sum_{i=1}^{\infty} n(i),$$

where $n(i)$ is the largest power of $p$ which divides $\alpha(i)$.

Definition 2.2. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ be sequences of integers with $\alpha_i \geq 2$ and $\beta_i \geq 2$ for each $i$. We say that $\alpha \leq \beta$ if $M_\alpha(p) \leq M_\beta(p)$ for all prime numbers $p$.

Definition 2.3. Let $X$ be a compact metric space. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a sequence of integers with $\alpha_i \geq 2$ for each $i$. We define $S_\alpha(X) = \{(x, f) \in X \times C(X, X) : \omega(x, f) \text{ is an } \alpha\text{-adic adding machine}\}$.

Definition 2.4. We call odrometers of type $\infty$ those $\alpha$ for which $M_\alpha(p) = \infty$ for all $p$, and we let $S_\infty(M) = \{(x, f) \in M \times C(M, M) : \omega(x, f) \text{ is an odometer of type } \infty\}$.

Theorem 2.5 ([4]). Let $\alpha, \beta \in (N \setminus \{1\})^N$, $m_i = \alpha(1)\alpha(2)\ldots\alpha(i)$, for each $i$, and $f : X \to X$ a continuous map of a compact topological space $X$. Then $f$ is topologically conjugate to $f_\alpha$ if and only if (1), (2), and (3) hold.

1. For each positive integer $i$, there is a cover $P_i$ of $X$ consisting of $m_i$ pairwise disjoint, nonempty, clopen sets which are cyclically permuted by $f$.

2. For each positive integer $i$, $P_{i+1}$ partitions $P_i$.

3. If $W_1 \supset W_2 \supset W_3 \supset \ldots$ is a nested sequence with $W_i \in P_i$ for each $i$, then $\cap_{i=1}^{\infty} W_i$ consists of a single point.

Moreover, in this case statement (4) also holds.

4. $X$ is metrizable and if $\text{mesh}(P_i)$ denotes the maximum diameter of an element of the cover $P_i$, then $\text{mesh}(P_i) \to 0$ as $i \to \infty$.

3 Our Results.

We show that there is a residual set of points $(x, f)$ in $M \times C(M, M)$ for which $\omega(x, f)$ is a particular $\alpha$-adic adding machine which we call an adding
machine of type $\infty$. What may be more surprising is that if $M$ has the fixed point property a generic element of $C(M, M)$ also generates uncountably many distinct $\alpha$-adic adding machines for every possible $\alpha$ [6].

In addition we prove the following [5]:

1. If $\alpha \neq \infty$, then $S_\alpha(M)$ is a nowhere dense subset of $M \times C(M, M)$ that contains no isolated points (and, in general, it does not need to be porous).

2. If $\alpha \leq \beta$, then $S_\alpha(M) \subseteq S_\beta(M)$.

Next, we state precisely the above mentioned results.

The next results have been obtained jointly with Udayan B. Darji ([6]).

**Theorem 3.1.** The set $S_\infty(M)$ is residual in $M \times C(M, M)$.

**Theorem 3.2.** Let $M$ be an $n$-manifold with the fixed point property. A generic $f \in C(M, M)$ has the property that for each $\alpha \in (N \setminus \{1\})^N$ there are continuum many pairwise disjoint $\omega$-limit sets of $f$ that are topologically conjugate to $\Delta_\alpha$.

The next results are obtained jointly with Paul D. Humke ([5]).

**Query.** What else can be said about $S_\alpha(X)$ when $\alpha$ is not of type $\infty$? How large is it?

**Remark 3.3.** In general $S_\alpha(X)$ does not need to be a closed set.

**Theorem 3.4.** Let $X$ be a compact metric space. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ such that $\alpha_i \geq 2$ for every $i \in N$. Then $S_\alpha(X)$ has no isolated points.

**Theorem 3.5.** Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ such that $\alpha_i \geq 2$ for every $i \in N$ and there exists a prime number $p$ with $M_\alpha(p) < \infty$. Then $S_\alpha(M)$ is a nowhere dense subset of $M \times C(M, M)$.

**Theorem 3.6.** If $\alpha \leq \beta$, then $S_\alpha(M) \subseteq S_\beta(M)$.

**Remark 3.7.** The set $S_\beta(M)$ may contain much more than just $\cup_{\alpha \leq \beta} S_\alpha(M)$.

**References**


