WEEK 2

LACEY ETZKORN, AMANDA TVEITE, AND STEPHEN CROUSER

Theorem. *Any countable set has a measure of zero (is null).*

Proof. Let A be any countable set. We will show that $\forall\ \epsilon > 0\ \exists\ I\ \ni A \subset I$. Let the n^{th} interval cover be equal to $[a_n - \frac{\epsilon}{2^{n+2}}, a_n + \frac{\epsilon}{2^{n+2}}]$ where $l(I_n) = \frac{\epsilon}{2^{n+1}}$. Thus, the length of the entire interval cover is given by:

$$l(I) = \sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2}$$

Since $A \subset I$ and the outer measure of an interval is it's length,

$$m(A) < m(I) = l(I) = \frac{\epsilon}{2} < \epsilon$$

□

Theorem. *A countable union of null sets is null.*

Proof. Let A be a set such that $A = \cup_{n=1}^{\infty} N_n$ where N_n is a countable set $\forall n \in \mathbb{N}$. We will show that $\forall\ \epsilon > 0\ \exists\ I\ \ni A \subset I$. Let the m^{th} interval cover of the n^{th} null set be equal to $[a_n^m - \frac{\epsilon}{2^{m+n+2}}, a_n^m + \frac{\epsilon}{2^{m+n+2}}]$ where $l(I_n^m) = \frac{\epsilon}{2^{m+n+1}}$. Thus, the length of the n^{th} interval cover is given by:

$$l(I_n) = \sum_{m=1}^{\infty} l(I_n^m) = \sum_{m=1}^{\infty} \frac{\epsilon}{2^{m+n+1}} = \frac{\epsilon}{2^{n+1}}$$

Summing the lengths of all the interval covers, we have:

$$l(I) = \sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2}$$

Since $A = \cup_{n=1}^{\infty} N_n \subset I$ and the outer measure of an interval is it's length,

$$m(A) < m(I) = l(I) = \frac{\epsilon}{2} < \epsilon$$

□

Theorem. *Outer measure is countably sub-additive and monotone.*

Theorem. *The outer measure of an interval is its length.*

Date: September 26, 2013.

Biggy Pairs Let Biggy Pairs be the set of all pairs of rational numbers. Then the sigma field of Biggy Pairs must be the set of sets of rationals with no elements, one element, two elements, three elements... up to a countably infinite number of rationals in one set. We can construct our set like this:

$$\begin{aligned}\{\emptyset\},\\ \{r_1\},\\ \{r_2\}, \{r_1, r_2\},\\ \{r_3\}, \{r_1, r_3\}, \{r_2, r_3\}, \{r_1, r_2, r_3\},\\ \{r_4\}, \{r_1, r_4\}, \{r_2, r_4\}, \{r_1, r_2, r_4\}, \{r_3, r_4\}, \{r_1, r_3, r_4\}, \{r_1, r_2, r_3, r_4\}\end{aligned}$$

Since the rationals are a countable set, we substituted writing out the rationals with lower case r, indexed with the natural numbers. It can be shown that there is a bijective function that relates $\mathbb{Q}$ with our set $\{r_n\}$. The set is constructed so that when a new number is introduced as a single-element set, all possible combinations of that number with the previous numbers are listed before moving onto the next single-element set. If we use $m \in \mathbb{N}$ to index the sets of the sigma field, we see that these single-element sets occur at the $1 + 2^{m-1}$th set. Since we hit every single element set, and take all unions/intersections of that element with previously used rationals before moving on to the next single element set, we can be sure that we have included all possible combinations of rationals of all possible set sizes in our sigma field.

Definition 1. *The set A is a **null set** if $\forall \epsilon > 0$ $\exists$ and interval cover I such that $A \subset I$ and $\sum_{n=1}^{\infty} l(I_n) < \epsilon$*

Definition 2. *The **outermeasure** of set A is $m^*(A) = \inf(Z_A)$ where*

$$Z_a = \{l(I) | A \subset I\}$$

Definition 3. *A set function is **monotone** when $A \subset B \Rightarrow f(A) < f(B)$*

Definition 4. *A collection of sets A is a **sigma-field** if:*

(1) $A_n \in \mathbb{R}$

(2) *M is closed under intersection and unions.*

(3) *M is closed under complementation.*

Definition 5. *A collection of sets A is **closed under countable unions and intersections** if $A_a, A_b \in A \Rightarrow A_a \cup A_b \in A$ and $A_a \cap A_b \in A$ for any pair $a, b \in \mathbb{N}$.*

Definition 6. *A collection of sets A is **closed under complementation** if $A_n \in A \Rightarrow A_n^c \in A$ $\forall$ $n \in \mathbb{N}$.*

Definition 7. *A set function $f(A)$ is **countably sub-additive** if:*

$$f(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} f(A_i)$$