The Kakeya Problem for Circular Arcs and the System of Graph-Null Sets

Miklós Laczkovich

By a classical result of Besicovitch (solving the so-called Kakeya problem), every line segment can be continuously moved in a plane set of arbitrarily small area such that it returns to its starting position with its direction reversed. Besicovitch's construction actually shows that every line segment can be moved to arrive at any prescribed position within a set of arbitrarily small area. F. Cunningham asked in 1974 whether or not a circular arc has a similar property. We present a partial solution found jointly with K. Héra showing that every circular arc shorter than the half circle has this property.

An attempt to solve the problem for circular arcs longer than the half circle led to the following observation: for every circle and for every $\varepsilon > 0$ there exists a function $f\colon [a,b] \to \mathbb{R}$ such that the area of the Minkowski sum of the circle and of the graph of f is less than ε. Sets with this property are called graph-null sets. We present some problems and recent results on graph-null sets.