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ON SOME PROPERTIES OF J-DENSITY TOPOLOGIES AND J-APPROXIMATELY CONTINUOUS FUNCTIONS

1 Introduction

Notations $\mathbb{R}$ - the set of real numbers,
$\mathbb{N}$ - the set of positive integers,
$|J|$ - the length of an interval J,
$\mathrm{diam}(A)$ - the diameter of the set $A \subset \mathbb{R}$
λ - the Lebesgue measure on $\mathbb{R}$,
$\mathcal{L}$ - the σ-algebra of Lebesgue measurable sets on $\mathbb{R}$,
$\mathcal{N}$ - the σ-ideal of Lebesgue null sets on $\mathbb{R}$,
$\mathcal{T}_{nat}$ - the natural topology on $\mathbb{R}$,
$A \triangle B = (A \setminus B) \cup (B \setminus A)$,
$J + x = \{a + x : a \in J\}$ for $J \subset \mathbb{R}$,
$J - x = \{a - x : a \in J\}$ for $J \subset \mathbb{R}$.

Let $A \in \mathcal{L}$ and $x_0 \in \mathbb{R}$.

We shall say that a point x_0 is a **density point** of A if

$$\lim_{h \to 0^+} \frac{\lambda((A - x_0) \cap [-h, h])}{2h} = 1 \Leftrightarrow$$
$$\lim_{n \to \infty} \frac{\lambda((A - x_0) \cap [-\frac{1}{n}, \frac{1}{n}])}{\frac{2}{n}} = 1.$$

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Let $\langle s\rangle = \{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and $s_n \nearrow \infty$.
We shall say that a point x_0 is an $\langle s\rangle$-**density point** of A if

$$\lim_{n\to\infty} \frac{\lambda((A-x_0)\cap[-\frac{1}{s_n},\frac{1}{s_n}])}{\frac{2}{s_n}} = 1.$$

2 A $\mathcal{J}$-density point

We say that a sequence $\mathcal{J} = \{J_n\}_{n\in\mathbb{N}}$ of closed intervals is **tending to zero** (denoted $J_n \underset{n\to\infty}{\longrightarrow} 0$) iff

$$\lim_{n\to\infty} \operatorname{diam}(J_n \cup \{0\}) = 0.$$

$\Im$ - the family of all sequences of closed intervals tending to zero.
Let $A \in \mathcal{L}$, $\mathcal{J} \in \Im$ and $x_0 \in \mathbb{R}$.
We shall say that a point x_0 is a $\mathcal{J}$-**density point** of the set A if

$$\lim_{n\to\infty} \frac{\lambda((A-x_0)\cap J_n)}{|J_n|} = 1.$$

An operator $\Phi_{\mathcal{J}}$
Let $A \in \mathcal{L}$ and $\mathcal{J} \in \Im$.

$$\Phi_{\mathcal{J}}(A) = \{x \in \mathbb{R} : x \text{ is a } \mathcal{J}\text{-density point of the set } A\}.$$

Theorem 1. *Let A, $B \in \mathcal{L}$ and $\mathcal{J} \in \Im$.*

1. $\Phi_{\mathcal{J}}(\emptyset) = \emptyset$ *and* $\Phi_{\mathcal{J}}(\mathbb{R}) = \mathbb{R}$,
2. $\Phi_{\mathcal{J}}(A \cap B) = \Phi_{\mathcal{J}}(A) \cap \Phi_{\mathcal{J}}(B)$,
3. *if* $(A \triangle B) \in \mathcal{N}$, *then* $\Phi_{\mathcal{J}}(A) = \Phi_{\mathcal{J}}(B)$,
4. $(\Phi_{\mathcal{J}}(A) \setminus A) \in \mathcal{N}$,
5. $\Phi_{\mathcal{J}}(A) \in F_{\delta\sigma}$.

Therefore $\Phi_{\mathcal{J}}$ is an almost lower density operator.

3 A $\mathcal{J}$-density topology

Theorem 2. *Let $\mathcal{J} \in \Im$. The family*

$$\mathcal{T}_{\mathcal{J}} = \{A \in \mathcal{L} : A \subset \Phi_{\mathcal{J}}(A)\}$$

is a topology stronger than the natural topology and it is called a $\mathcal{J}$-density topology.

$\Im_\alpha$ - a some special subset of $\Im$
Let

$$\alpha(\mathcal{J}) = \limsup_{n\to\infty} \frac{\operatorname{diam}(J_n \cup \{0\})}{|J_n|} < \infty.$$

Put

$$\Im_\alpha = \{\mathcal{J} \in \Im : \alpha(\mathcal{J}) < \infty\}.$$

If $\mathcal{J} \in \Im_\alpha$, then we have

1. $(\Phi_{\mathcal{J}}(A) \triangle A) \in \mathcal{N}$,
2. $\Phi_d(A) \subset \Phi_{\mathcal{J}}(A)$, where $\Phi_d(A)$ is the set of all density points of A.

Theorem 3. *If $\mathcal{J} \in \Im_\alpha$, then the topology $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is von-Neuman topology associated with the Lebesgue measure. It means that:*

1. *$(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is Baire space;*
2. *$(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ satisfies ccc;*
3. *$\mathbb{K}(\mathcal{T}_{\mathcal{J}}) = \{A \in \mathcal{L} \colon \lambda(A) = 0\}$;*
4. *$\mathcal{B}a(\mathcal{T}_{\mathcal{J}}) = \mathcal{L}$.*

Theorem 4. *For every sequence of intervals $\mathcal{J} \in \Im_\alpha$ the following conditions are equivalent:*

i) $\mathcal{J} \in \Im_\alpha$;

ii) $\mathcal{T}_d \subset \mathcal{T}_{\mathcal{J}}$, where $\mathcal{T}_d$ is a classical density topology .

Let $\mathcal{J} \in \Im_\alpha$. We have the following properties:

(a) $A \in \mathcal{N}$ if and only if A is a closed and discrete set with respect to a topology $\mathcal{T}_{\mathcal{J}}$;

(b) a set $A \subset \mathbb{R}$ is compact with respect to a topology $\mathcal{T}_{\mathcal{J}}$ if and only if A is finite;

(c) $\mathcal{N} = \mathbb{K}(\mathcal{T}_{\mathcal{J}})$ and $\mathcal{L} = \mathcal{B}a(\mathcal{T}_{\mathcal{J}})$;

(d) $\mathcal{B}(\mathcal{T}_{\mathcal{J}}) = \mathcal{L}$;

(e) $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is neither first countable, nor second countable, nor separable, nor Lindelöf space.

Theorem 5. *For every sequence of intervals $J = \{J_n\}_{n\in\mathbb{N}} \in \Im$ there exists a sequence of intervals $K = \{K_n\}_{n\in\mathbb{N}} \in \Im$ such that*

$$\mathcal{T}_{\mathcal{J}} \setminus \mathcal{T}_K \neq \emptyset \text{ and } \mathcal{T}_K \setminus \mathcal{T}_{\mathcal{J}} \neq \emptyset.$$

4 A $\mathcal{J}$-approximately continuous function

Let $f : \mathbb{R} \to \mathbb{R}$ and $\mathcal{J} \in \Im$.

We shall say that f is **$\mathcal{J}$-approximately continuous** at a point $x_0 \in \mathbb{R}$ if there exists a set $A_{x_0} \in \mathcal{L}$ such that

$$x_0 \in \Phi_{\mathcal{J}}(A_{x_0}) \text{ and } f(x_0) = \lim_{\substack{x\to x_0,\\ x\in A_{x_0}}} f(x).$$

Theorem 6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function and let $\mathcal{J} \in \Im$. The function f is $\mathcal{J}$-approximately continuous at a point $x_0 \in \mathbb{R}$ if and only if the following condition is fulfilled:*

$$\underset{\varepsilon>0}{\forall}\ x_0 \in \Phi_{\mathcal{J}}(\{x \in \mathbb{R} : \ |f(x) - f(x_0)| < \varepsilon\}).$$

Theorem 7. *A function $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable if and only if there exists a sequence $\mathcal{J} \in \Im$ and a set $B_{\mathcal{J}} \in \mathcal{N}$ such that the function f is $\mathcal{J}$-approximately continuous at each point $x \in \mathbb{R} \setminus B_{\mathcal{J}}$.*

Theorem 8. *Let $f : \mathbb{R} \to \mathbb{R}$ and $\mathcal{J} \in \Im$. The function f is $\mathcal{J}$-approximately continuous if and only if for any $a \in \mathbb{R}$ the sets $E_a = \{x \in \mathbb{R} : \ f(x) < a\}$ and $E^a = \{x \in \mathbb{R} : \ f(x) > a\}$ belong to the topology $\mathcal{T}_{\mathcal{J}}$.*

Theorem 9. *Let $f : \mathbb{R} \to \mathbb{R}$ and $\mathcal{J} \in \Im$. The function f is $\mathcal{J}$-approximately continuous if and only if $f \in \mathcal{C}_{\mathcal{T}_{nat}}^{\mathcal{T}_{\mathcal{J}}}$.*

$\mathcal{C}_{\mathcal{T}_{nat}}^{\mathcal{T}_{\mathcal{J}}}$ is the family of all continuous functions $f : (\mathbb{R}, \mathcal{T}_{\mathcal{J}}) \to (\mathbb{R}, \mathcal{T}_{nat})$.

Theorem 10. *Let $\mathcal{J} \in \Im$. If $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{J}$-approximately continuous function, then f belongs to the first class of Baire.*

5 The separation axioms

Theorem 11. *For any $\mathcal{J} \in \Im$ the space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is a Hausdorff space.*

Theorem 12. *For any $\mathcal{J} \in \Im_\alpha$ the space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is regular.*

Theorem 13 (An analogue of Lusin-Menchoff Theorem)**.** *Let $\mathcal{J} \in \Im_\alpha$ and E be a Lebesgue measurable set. Then for every $\mathcal{T}_{nat}$-closed set X such that $X \subset (E \cap \Phi_{\mathcal{J}}(E))$, there exists a $\mathcal{T}_{nat}$-perfect set P such that $X \subset P \subset E$ and $X \subset \Phi_{\mathcal{J}}(P)$.*

Theorem 14. *Let $\mathcal{J} \in \Im_\alpha$ and E is a F_σ set such that $E \subset \Phi_{\mathcal{J}}(E))$. Then there exists a $\mathcal{J}$-approximately continuous function f such that*

1) $0 < f(x) \leq 1$ *for* $x \in E$*,*

2) $f(x) = 0$ *for* $x \notin E$*.*

Theorem 15. *For every $\mathcal{J} \in \Im_\alpha$ the space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is completely regular.*

Theorem 16. *For any $\mathcal{J} \in \Im$ the space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ is not normal.*

Problem 17. *Is the space $(\mathbb{R}, \mathcal{T}_{\mathcal{J}})$ regular (completely regular) for every sequence of intervals $\mathcal{J} \in \mathcal{J}$?*

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