Real Analysis Exchange
Summer Symposium XXXX, 2016, pp. 1–2

Martin Koc,* RSJ a.s., Na Florenci 2116/15, 110 00 Praha 1, Czech Republic. email: martin.koc@rsj.com

EXTENSIONS OF VECTOR-VALUED FUNCTIONS WITH PRESERVATION OF DERIVATIVES

Let X and Y be Banach or normed linear spaces and $F \subset X$ a closed set. We apply our recent extension theorem for vector-valued Baire one functions [KK, Theorem 1.1] to obtain an extension theorem for vector-valued functions $f\colon F \to Y$ with pre-assigned derivatives, with preservation of differentiability (at every point where the pre-assigned derivative is actually a derivative), preservation of continuity, preservation of (pointwise) Lipschitz property and (for finite dimensional domain X) preservation of strict differentiability and global (eventually local) Lipschitz continuity. It is a joint work with Jan Kolář. Our results can be roughly viewed as a joint generalization of extension theorems of Tietze-Dugundji, Whitney (C^1-case) and McShane-Johnson-Lindenstrauss-Schechtman (see [JLS, Theorem 2]), all in pointwise fashion. Nevertheless, they were created as vector-valued generalizations of extension theorems of Aversa, Laczkovich, Preiss [ALP] and Koc, Zajíček [KZ].

Theorem 1. *Let X, Y be normed linear spaces, $F \subset X$ a closed set, $f\colon F \to Y$ an arbitrary function and $L\colon F \to \mathcal{L}(X,Y)$ a Baire one function. Let $p \in \mathbb{N} \cup \{\infty\}$. Then there exists a function $\bar{f}\colon X \to Y$ such that*

(i) $\bar{f} = f$ *on* F,

(ii) *if $a \in F$ and f is continuous at a (with respect to F), then $\bar{f}$ is continuous at a,*

(iii) *if $a \in F$, $\alpha \in (0,1]$ and f is α-Hölder continuous at a (with respect to F), then $\bar{f}$ is α-Hölder continuous at a; in particular, if f is Lipschitz at a (with respect to F), then $\bar{f}$ is Lipschitz at a,*

Mathematical Reviews subject classification: Primary: 26B05, 26B35, 54C20, 26B12
Key words: vector-valued differentiable functions, extensions, strict differentiability, partitions of unity
*The author would like to thank the private company RSJ a.s. for the support of his research activities.

(iv) if $a \in F$ and $L(a)$ is a relative Fréchet derivative of f at a (with respect to F), then $(\bar{f})'(a) = L(a)$,

(v) $\bar{f}$ is continuous on $X \setminus F$,

(vi) if X admits C^p-smooth partition of unity, then $\bar{f} \in C^p(X \setminus F, Y)$.

Moreover, if $\dim X < \infty$, then

(vii) if $a \in F$, L is continuous at a and $L(a)$ is a relative strict derivative of f at a (with respect to F), then the Fréchet derivative $(\bar{f})'$ is continuous at a with respect to $(X \setminus F) \cup \{a\}$ and $L(a)$ is the strict derivative of $\bar{f}$ at a (with respect to X),

(viii) if $a \in F$, $R > 0$, L is bounded on $B(a, R) \cap F$ and f is Lipschitz continuous on $B(a, R) \cap F$, then $\bar{f}$ is Lipschitz continuous on $B(a, r)$ for every $r < R$; if L is bounded on F and f is Lipschitz continuous on F, then $\bar{f}$ is Lipschitz continuous on X.

References

[1] V. Aversa, M. Laczkovich, D. Preiss, *Extension of differentiable functions*, Comment Math. Univ. Carolin. 26 (1985) 597–609.

[2] W.B. Johnson, J. Lindenstrauss, G. Schechtman, *Extensions of Lipschitz maps into Banach spaces*, Israel Journal of Mathematics 54:2 (1986) 129–138.

[3] M. Koc, J. Kolář, *Extensions of vector-valued Baire one functions with preservation of points of continuity*, J. Math. Anal. Appl. 442:1 (2016) 138–148.

[4] M. Koc, J. Kolář, *Extensions of vector-valued functions with preservation of derivatives*, arXiv:1602.05750v1 [math.CA] (`http://arxiv.org/abs/1602.05750v1`).

[5] M. Koc, L. Zajíček, *A joint generalization of Whitney's $\mathcal{C}^1$ extension theorem and Aversa-Laczkovich-Preiss' extension theorem*, J. Math. Anal. Appl. 388 (2012) 1027–1039.