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DIFFERENTIABLE EXTENSIONS FROM A CLOSED SET

Let us recall a result from the talk of Martin Koc on extensions of differentiable functions defined on a closed set.

Corollary (Koc, Kolář [4, Corollary 4.4/5])**.** *Let Y be a normed linear space, $H \subset \mathbb{R}^n$ a closed set, $f\colon H \to Y$ a function and $L\colon H \to \mathcal{L}(\mathbb{R}^n, Y)$ a relative derivative of f (on H). Assume that L is Baire one on H. Then there is $\bar f\colon \mathbb{R}^n \to Y$ such that*

(1) $\bar f$ is differentiable on $\mathbb{R}^n$,

(2) $\bar f = f$ and $(\bar f)' = L$ on H,

(3) if $a \in H$, L is continuous at a and $L(a)$ is a (relative) strict derivative of f at a then $(\bar f)'$ is continuous at a,

(4) f is $\mathcal{C}^\infty$ on $\mathbb{R}^n \setminus H$.

Note that a candidate derivative L is required and assumed to be Baire one. Here we present results where this requirement is removed and replaced by conditions on the set H. This goal is motivated by Section 4 of [5].

Definition 1 (Contingent and paratingent)**.** For $H \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, define

$$\begin{aligned}\operatorname{Tan}(H, x) = \{v \in \mathbb{R}^n : \exists x_k \in H,\ \alpha_k \in [0, \infty),\ x_k \to x,\\ \text{such that } \alpha_k\,(x_k - x) \to v\},\end{aligned}$$

the *contingent cone* (sometimes also the *tangent cone*) of H at x, and

$$\begin{aligned}\operatorname{Ptg}(H, x) = \{v \in \mathbb{R}^n : \exists x_k, y_k \in H,\ \alpha_k \in \mathbb{R},\ x_k \to x,\ y_k \to x,\\ \text{such that } \alpha_k\,(y_k - x_k) \to v\},\end{aligned}$$

the *paratingent cone* of H at x.

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Note that $\operatorname{Tan}(H,x) \subset \operatorname{Ptg}(H,x)$. Let $\operatorname{der} H$ denote the set of accumulation points of H.

Definition 2. Let H be a subset of $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ let

$$a_H(x) = \sup\big\{|\det(v_1,\dots,v_n)| : v_1,\dots,v_n \text{ unit vectors from } \operatorname{Tan}(H,x)\big\},$$
$$p_H(x) = \sup\big\{|\det(v_1,\dots,v_n)| : v_1,\dots,v_n \text{ unit vectors from } \operatorname{Ptg}(H,x)\big\}.$$

The following simply formulated result already generalizes [5, Proposition 4.10]: $\operatorname{Tan}(H,x)$ is replaced by a larger set $\operatorname{Ptg}(H,x)$. Moreover, vector-valued mappings are allowed.

Proposition 3. *Let $H \subset \mathbb{R}^n$ be a nonempty closed set such that $\operatorname{Ptg}(H,x)$ spans $\mathbb{R}^n$ for every $x \in \operatorname{der} H$. Let Y be a normed linear space and $f\colon H \to Y$ a function (relatively) strictly differentiable at every $x \in \operatorname{der} H$. Then there exists a differentiable extension of f defined on $\mathbb{R}^n$.*

Proposition 4. *Under the assumptions of the previous Proposition, there exists a differentiable extension $\bar{f}\colon \mathbb{R}^n \to Y$ of f (which is C^∞ on $\mathbb{R}^n \setminus H$) such that:*
$\bar{f}$ is strictly differentiable at x (with respect to $\mathbb{R}^n$) and the derivative of $\bar{f}$ is continuous at x (with respect to $\mathbb{R}^n$) <u>for</u>

(a) all $x \in \mathbb{R}^n \setminus \operatorname{der} H$ and

(b) all $x \in \operatorname{der} H$ where the (unique) relative strict derivative of f with respect to $\operatorname{der} H$ is continuous,

(c) and hence, in particular, for all $x \in \operatorname{der} H$ such that there is $r_x > 0$ and $p_x > 0$ such that $p_H(z) \ge p_x$ for all $z \in B(x,r_x) \cap \operatorname{der} H$.

Proposition 5 (Vector generalization of [5, Corollary 4.3])**.** *Let $H \subset \mathbb{R}^n$ be a nonempty closed set. Assume $\operatorname{der} H = \bigcup_{m\in\mathbb{N}} D_m$ where, for each $m \in \mathbb{N}$, D_m is a closed subset of H and there is a positive number a_m such that $a_H(x) \ge a_m$ for every $x \in D_m$.*

Let Y be a normed linear space, $f\colon H \to Y$ a function. Assume that, for every $x \in \operatorname{der} H$, f is (relatively) differentiable at x.

Then there exists a differentiable extension $\bar{f}\colon \mathbb{R}^n \to Y$ of f.

Proposition 6 (a generalization of [5, Theorem 4.6] and [5, Corollary 4.7])**.** *Let $H \subset \mathbb{R}^n$ be a nonempty closed set satisfying the following condition which is equivalent to* (C) *in [5, p. 1035]:*

$$\text{(C)} \qquad \inf\left\{a_H(y) : y \in (\operatorname{der} H) \cap \overline{B(0,R)}\right\} > 0 \qquad \textit{for every } R \in (1,\infty).$$

Let Y be a normed linear space, $f\colon H \to Y$ a function and assume that the relative derivative of f at x exists for every $x \in \operatorname{der} H$. Then there is a differentiable extension $\bar f\colon \mathbb{R}^n \to Y$ of f such that

(a) $\bar f$ is C^∞ on $\mathbb{R}^n \setminus H$ and C^1 on $\mathbb{R}^n \setminus \operatorname{der} H$,

(b) $\bar f$ is strictly differentiable at x and the derivative of $\bar f$ is continuous at x for all $x \in \operatorname{der} H$ where f is relatively strictly differentiable.

Proposition 7 (another generalization of [5, Corollary 4.7])**.** *Let $H \subset \mathbb{R}^n$ be a nonempty closed set, Y a normed linear space, $f\colon H \to Y$ a function and assume that the relative strict derivative of f at x exists for every $x \in \operatorname{der} H$. Moreover, let the following condition hold:*

$$(\mathrm{C}^{\mathrm{Ptg}}) \qquad \inf\left\{p_H(y) : y \in (\operatorname{der} H) \cap \overline{B(0,R)}\right\} > 0 \qquad \text{for every } R \in (1,\infty).$$

Then there is an extension $\bar f\colon \mathbb{R}^n \to Y$ of f that is C^1 on $\mathbb{R}^n$ and C^∞ on $\mathbb{R}^n \setminus H$.

Proposition 8 (a joint generalization of Proposition 6 and Proposition 7)**.** *Let $H \subset \mathbb{R}^n$ be a nonempty closed set, and $S \subset \operatorname{der} H$. Assume that,*

$$(\mathrm{C}^*) \qquad \begin{aligned} &\text{for every } x \in \operatorname{der} H, \text{ there exists } r_x, d_x > 0 \text{ such that}\\ &\qquad p_H(y) \ge d_x \text{ for every } y \in B(x,r_x) \cap S \text{ and}\\ &\qquad a_H(y) \ge d_x \text{ for every } y \in B(x,r_x) \cap (\operatorname{der} H) \setminus S. \end{aligned}$$

Let Y be a normed linear space, $f\colon H \to Y$ a function and assume that, for every $x \in (\operatorname{der} H) \setminus S$, f is relatively differentiable at x and, for every $x \in S$, f is relatively strictly differentiable at x. Then there is a differentiable extension $\bar f\colon \mathbb{R}^n \to Y$ of f such that

(a) $\bar f$ is C^∞ on $\mathbb{R}^n \setminus H$ and C^1 on $\mathbb{R}^n \setminus \operatorname{der} H$,

(b) $\bar f$ is strictly differentiable at x and the derivative of $\bar f$ is continuous at x for every $x \in S$.

Corollary 9 (C^1-case of Whitney's theorem)**.** *Let $H \subset \mathbb{R}^n$ be a nonempty closed set, Y a normed linear space, $f\colon H \to Y$ a function, $E \subset H$ a set that contains $\operatorname{der} H$.*

Let $L\colon E \to \mathcal{L}(\mathbb{R}^n, Y)$ be given such that, for every $x \in E$, $L(x)$ is a (relative) strict derivative of f at x (with respect to H). Assume L is continuous.

Then there is an extension $\bar f\colon \mathbb{R}^n \to Y$ of f that is C^1 on $\mathbb{R}^n$ and C^∞ on $\mathbb{R}^n \setminus H$ such that $(\bar f)'(x) = L(x)$ for every $x \in E$.

Theorem 10. *Let $H \subset \mathbb{R}^n$ be a nonempty closed set. Assume* $\operatorname{der} H \subset E := \bigcup_{m\in\mathbb{N}} E_m$, *where, for each $m \in \mathbb{N}$, E_m is closed, $E_m = W_m \cup D_m \cup S_m$. Assume*

(1) for every m, there is $a_m > 0$ such that $a_H(x) \geq a_m$ for every $x \in D_m$.

(2) $\operatorname{Ptg}(H, x)$ spans $\mathbb{R}^n$ for every $x \in \bigcup_{m\in N} S_m$.

Let Y a normed linear space, $f \colon H \to Y$ a function. Let $L \colon E \to \mathcal{L}(\mathbb{R}^n, Y)$ be defined as follows: For $x \in \bigcup_m D_m$, $L(x)$ is a (relative) derivative of f at x with respect to H and, for $x \in \bigcup_m S_m$, $L(x)$ is a (relative) strict derivative of f at x with respect to H. (The derivatives are unique if they exist.)
For $x \in E \setminus \bigcup_m (D_m \cup S_m)$, let $L(x)$ be an arbitrary (relative) derivative of f at x with respect to H. Also assume, for every $m \in \mathbb{N}$ and $x \in W_m$, that $L|_{E_m}$ is continuous at x.

Then there exists a differentiable extension $\bar{f} \colon \mathbb{R}^n \to Y$ of f such that

(a) $(\bar{f})'(x) = L(x)$ for every $x \in E$.

(b) $\bar{f}$ is C^∞ on $\mathbb{R}^n \setminus H$ and C^1 on $\mathbb{R}^n \setminus \operatorname{der} H$,

(c) $\bar{f}$ is strictly differentiable at x (with respect to $\mathbb{R}^n$) and the derivative of $\bar{f}$ is continuous at x (with respect to $\mathbb{R}^n$) for all $x \in \operatorname{der} H$ such that $L(x)$ is a (relative) strict derivative of f (with respect to H) and L is continuous at x (with respect to E),

(d) in particular, $\bar{f}$ is strictly differentiable at x (with respect to $\mathbb{R}^n$) and the derivative of $\bar{f}$ is continuous at x (with respect to $\mathbb{R}^n$) for all $x \in \operatorname{der} H$ such that

(i) $L(x)$ is a (relative) strict derivative of f (with respect to H),

(ii) L is continuous at x with respect to $\{x\} \cup \bigcup_{m\in\mathbb{N}} W_m$ and

(iii) there exists $r_x > 0$, $a_x > 0$ and $p_x > 0$ such that we have
$a_H(z) \geq a_x$ for every $z \in \bigcup_{m\in\mathbb{N}} D_m \cap B(x, r_x)$, and
$p_H(z) \geq p_x$ for every $z \in \bigcup_{m\in\mathbb{N}} S_m \cap B(x, r_x)$,

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