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OPENNESS AND WEAK OPENNESS OF MULTIPLICATION IN THE SPACE OF FUNCTIONS OF BOUNDED VARIATION

Abstract Let $C[0,1]$ be the space of all continuous real-valued functions defined on $[0,1]$ with the supremum norm $\|f\| = \sup_{t \in [0,1]} |f(t)|$. There are some natural operations on $C[0,1]$, for example, addition, multiplication, minimum and maximum. In [4, 10, 11] such operations were investigated. All the operations are continuous but only addition, minimum and maximum are open as mappings from $C[0,1] \times C[0,1]$ to $C[0,1]$.

Remark 1 (Fremlin's example). *[4] In 2004, D. H. Fremlin observed that for $f \colon [0,1] \to \mathbb{R}$, $f(x) = x - \frac{1}{2}$, one has $f^2 \in B^2(f, \frac{1}{2}) \setminus \operatorname{Int} B^2(f, \frac{1}{2})$. Hence multiplication is not an open mapping from $C[0,1] \times C[0,1]$ into $C[0,1]$.*

Definition 2. *[4, 2] A map between topological spaces is weakly open if the image of every non-empty open set has a non-empty interior.*

In [4] it is shown that the multiplication in $C[0,1]$ is a weakly open operation. This was generalized in [8] for $C(0,1)$ and in [2] for $C(X)$, where X is an arbitrary interval.

In [7, 8] there are considered some properties of multiplication and others operations in the algebra $C(X)$ of real-valued continuous functions defined on a compact topological space X. Properties of the product of open balls and of n open sets in the space of continuous functions on $[0,1]$ are studied in [5] and [6], respectively.

There is an increasing interest in the study of concepts related to the openness and weak openness of natural bilinear maps on certain function spaces. The reason of it may be that the classical Banach open mapping principle is not true for bilinear maps. In [1], the authors show that multiplication from $L^p(X) \times L^q(X)$ onto $L^1(X)$, where (X, μ) is an arbitrary measure space and $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq \infty$, is an open mapping.

We study problem, stated in [3, Question 18.24 and 18.25], of openness and week openness in the space $BV[0,1]$ of functions of bounded variation and in the space $CBV[0,1]$ of continuous functions of bounded variation, both defined on $[0,1]$.

There are two natural norms in $BV[0,1]$ and $CBV[0,1]$: the supremum norm $\|f\| = \sup_{t \in [0,1]} |f(t)|$ and the norm defined by variation $\|f\|_{BV} = |f(0)| + V_0^1(f)$, where $V_a^b(f) = \sup_{a = t_0 < t_1 < \ldots < t_n = b} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$ is the variation of f on $[a,b]$. It is worth to mention that $(BV[0,1], \| \ \|_{BV})$ and $(CBV[0,1], \| \ \|_{BV})$ are complete, whereas $(BV, \| \ \|)$ and $(CBV, \| \ \|)$ are not.

Here are the most important results obtained and unresolved problem.

Theorem 3. *Multiplication is a weakly open mapping in $\big(CBV[0,1], \| \ \|_{BV}\big)$.*

Theorem 4. *Multiplying is a weakly open mapping in $(CBV[0,1], \| \ \|)$.*

Theorem 5. *Multiplying is a weakly open mapping in $(BV[0,1], \| \ \|_{BV})$.*

Question. *Does the multiplication in $(BV[0,1], \| \ \|_{BV})$ is an open operation?*

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