Real Analysis Exchange
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ACHIEVEMENT SETS OF CONDITIONALLY CONVERGENT SERIES

We call the set $A(x_n) = \{\sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n) \in \{0,1\}^{\mathbb{N}}\}$ *the set of subsums* or *the achievement set*. By $SR(x_n) = \{\sum_{n=1}^{\infty} x_{\sigma(n)} : \sigma \in S_{\infty}\}$ we denote the sum range. Due to Guthrie, Nymann and Saenz we know that the achievement set of an absolutely summable sequence of reals can be a finite set, a finite union of intervals, homeomorphic to the Cantor set or it can be a so called Cantorval. A Cantorval is a set homeomorphic to the union of the Cantor set and sets which are removed from the unite segment by even steps of the Cantor set construction.

Theorem 1. *For sequences of reals with* $\lim_{n\to\infty} x_n = 0$ *we have:*

- *A series* $\sum_{n=1}^{\infty} x_n$ *is potentially conditionally convergent (both series of positive and negative terms are divergent) if and only if* $A(x_n) = \mathbb{R}$.
- *A series of negative terms is convergent and a series of positive terms is divergent (or vice versa) if and only if the achievement set of* (x_n) *is a half line.*

Considering the sets of subsums of series (or achievement sets) we show that for conditionally convergent series the multidimensional case is much more complicated than that of the real line and we are still far from the full topological classification of such sets. Many surprising examples are presented and the ideas standing behind them are catched in general theorems. We observe among others that for the achievement set $A(x_n)$ of conditionally convergent series in $\mathbb{R}^2$ the following are possible.

- The intersection of $A(x_n)$ and $SR(x_n)$ could be a singleton and moreover we mention that it is always nonempty set;
- $A(x_n)$ can be a graph of function;
- $A(x_n)$ can be a dense set in $\mathbb{R}^2$ with an empty interior;
- $A(x_n)$ can be neither F_σ nor G_δ-set;
- $A(x_n)$ can be an open set not equal to the whole $\mathbb{R}^2$;