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DUAL OF THE SPACE OF HK_r INTEGRABLE FUNCTIONS 1/11

We define for $1 \leq r < \infty$ a norm for the class HK_r of functions which are Henstock-Kurzweil integrable in L^r sense. We then establish that the dual in this norm is isometrically isomorphic to $L^{r'}$ and is therefore a Banach space, and in the case $r = 2$, a Hilbert space. We give results pertaining to convergence and weak convergence in this space. Finally we discuss the completion of HK_r in term of distributional derivatives.

1 History and Aim

The idea of integration as a means for recovering a function from its derivative goes back to the earliest days of calculus. Denjoy (1912) and Perron (1914) developed nonabsolute integration methods that recover a function from its classical derivative. Later, Burkill (1931) developed a Perron-type integration process for recovering a function from its approximate derivative. Working independently, Kurzweil and Henstock developed an integration process equivalent to those of Denjoy and Perron, but which maintains the sense of Riemann integration by defining the integral as the limit of Riemann sums, subject to a pointwise-defined positive gauge function. This integration process has found to be suitable for many applications, for example in Haar and Walsh series [11] and zero-dimensional groups [12], [13], [14] and [15].

In his well-known paper [1], Alexiewicz defined the following norm on the space of Denjoy integrable functions f:

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$$\|f\|_A = \max\{F(x) : x \in [a,b]\}$$

where F is given by the indefinite Denjoy primitive of f:

$$F(x) = (D)\int_a^x f(t)\,dt.$$

Since the Denjoy and Henstock-Kurzweil integrals are equivalent (see [6]) we have that

$$F(x) = (HK)\int_a^x f(t)\,dt.$$

Alexiewicz showed that the dual of this space is isomorphic to the space of functions of bounded variation on $[a,b]$.

Bongiorno and Panchapagesan [2] and Talvila [16] extended Alexiewicz's work by describing the completion of the space of Denjoy integrable functions with respect to the Alexiewicz norm.

In 1961, Calderon and Zygmund [4] described the L^r derivative, which is preserved at individual points under operations such as fractional integration and singular integral transformations. In 1967, L. Gordon [5] described the P^r integral, a Perron type integral that recovers a function from its L^r derivative. In 2004 Musial and Sagher [7] developed a Henstock-Kurzweil type integral, (HK_r), that also recovers a function from its L^r derivative. Indeed, the HK_r integral extends the P^r integral, though it is not known whether the P^r integral integrates all HK_r-integrable functions. In 2015, P. Musial and F. Tulone gave an integration by parts formula for the HK_r integral.

In this paper, the authors describe a norm on the space of HK_r-integrable functions, as well as the dual and completion of this space.

2 Preliminaries

We begin with several definitions. We will assume the following:

- If an integral is not specified, it is a Lebesgue integral.
- We will assume $f : [a,b] \to R$ and $1 \le r < \infty$.
- A gauge is a positive function defined on $[a,b]$.
- A tagged interval is an ordered pair $(x, [c,d])$ such that $x \in [c,d]$.
- If δ is a gauge, a tagged interval $(x, [c,d])$ is said to be δ-fine if $[c,d] \subseteq (x-\delta(x), x+\delta(x))$.

Definition 2.1. *[7] A real-valued function $f : [a, b] \to R$ is L^r Henstock-Kurzweil integrable ($f \in HK_r [a, b]$) if there exists a function $F \in L^r [a, b]$ so that for any $\varepsilon > 0$ there exists a gauge function $\delta (x) > 0$ so that whenever $\mathcal{P} = \left\{ (x_i, [c_i, d_i])_{1 \le i \le q} \right\}$ is a δ-fine tagged partition of $[a, b]$ we have*

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} (L) \int_{c_i}^{d_i} |F (y) - F (x_i) - f (x_i) (y - x_i)|^r \, dy \right)^{\frac{1}{r}} < \varepsilon. \tag{2.1}$$

We will say that a gauge δ is HK_r-*appropriate* for ε and for f if (2.1) holds for any δ-fine tagged partition $\mathcal{P}$.

It is clear that if $f \in HK_r[a, b]$ then for any $[c, d] \subset [a, b], f \in HK_r[c, d]$. It is shown in [7] that if f is HK_r-integrable on $[a, b]$, the following function is well-defined for all $x \in [a, b]$:

$$F (x) = (HK_r) \int_{a}^{x} f (t) \, dt \tag{2.2}$$

and we say F is the indefinite HK_r primitive of f.

We recall the definition of the L^r derivative given by Calderon and Zygmund in [4].

Definition 2.2. *[4] For $1 \le r < \infty$, a function $F \in L^r ([a, b])$ is said to be L^r-differentiable at $x \in [a, b]$ if there exists $a \in \mathbb{R}$ such that*

$$\int_{-h}^{h} |F (x + t) - F (x) - at|^r \, dt = o \left(h^{r+1} \right).$$

In this case we write

$$F_r'(x) = a$$

Definition 2.3. *[7] We say that F is absolutely continuous in L^r sense on E, and write $F \in AC_r (E)$, if for all $\varepsilon > 0$ there exist $\eta > 0$ and a gauge function $\delta (x)$ defined on E so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\}$ is a finite collection of non-overlapping δ-fine tagged intervals having tags in E and satisfying*

$$\sum_{i=1}^{q} (d_i - c_i) < \eta$$

then

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F (y) - F (x_i)|^r \, dy \right)^{\frac{1}{r}} < \varepsilon.$$

Definition 2.4. *[7] We say that F is generalized absolutely continuous in L^r sense on E, and write $F \in ACG_r(E)$, if E can be written*

$$E = \bigcup_{i=1}^{\infty} E_i$$

and $F \in AC_r(E_i)$ for all i.

Theorem 2.5. *[7] Let $1 \le r < \infty$. A function f is HK_r-integrable on $[a,b]$ if and only if there exists a function $F \in ACG_r([a,b])$ so that $F_r^{'} = f$ a.e.*

3 HK_r-Duality

The results of this section will appear in [8].

We now define a norm on the space of HK_r-integrable functions.

Definition 3.1. *Let $f \in HK_r([a,b])$. We define the HK_r norm of f as follows:*

$$\|f\|_{HK_r} := \|F\|_{L^r}.$$

where F is the indefinite HK_r primitive of f as defined in (2.2).

It can easily be shown that the space HK_r is separable.

Definition 3.2. *Suppose $\{f_n\}$ is a sequence of functions in $HK_r([a,b])$ having indefinite HK_r primitives $\{F_n\}$. We will say $\{f_n\}$ converges (strongly) in HK_r sense to $f \in HK_r([a,b])$ if $\{F_n\}$ converges to F in L^r, where F is the HK_r primitive of f.*

We identify a function $f \in HK_r$ with the equivalence class of real-valued functions that are equal to f a.e. Therefore we have established an isometric one-to-one correspondence between the space HK_r and the space of HK_r primitives.

We note that $HK_r([a,b])$ fails to be complete with respect to the $\|\cdot\|_{HK_r}$ norm.

Now we will discuss criteria so that a sequence $\{f_n\}$ of functions in the space HK_r which converges pointwise to a function $f \in HK_r$ also converges strongly in the HK_r norm to f.

Definition 3.3. *Let $\{f_n\}$ be a sequence in $HK_r([a,b])$. We say that $\{f_n\}$ is uniformly HK_r-integrable, and write $\{f_n\} \in UHK_r([a,b])$, if for each $\varepsilon > 0$ there exists a gauge function which is HK_r-appropriate for ε and for f_n, $\forall n \ge 1$.*

Theorem 3.4. *Let $\{f_n\} \in UHK_r([a,b])$ and suppose that $\{f_n\}$ converges uniformly to a function f. For each n let F_n be as in (2.2). Then*

1. *$\{F_n\}$ converges uniformly to a function F,*

2. *$f \in HK_r([a,b])$ and F is the HK_r primitive of f, and*

3. *$\{f_n\}$ converges to f strongly in HK_r.*

We use the Riesz Representation Theorem to characterize the dual to HK_r.

Theorem 3.5. *Let $1 \leq r < \infty$. ϕ is a bounded linear functional on the space HK_r if and only if there exists an absolutely continuous function $\widehat{G}$ defined on $[a,b]$ so that $\widehat{G}(b) = 0$ and $\left(\widehat{G}\right)' \in L^{r'}([a,b])$, where $r' = r/(r-1)$, so that*

$$\phi(f) = (HK_r) \int_a^b f(x)\, \widehat{G}(x)\, dx. \tag{3.1}$$

In this case,

$$\|\phi\|_{HK_r} = \left\| \left(\widehat{G}\right)' \right\|_{L^{r'}}.$$

We now characterize weak convergence of a sequence in HK_r.

Corollary 3.6. *A sequence of functions $\{f_n\}$ in HK_r converges weakly to a function $f \in HK_r$ if and only if for every absolutely continuous function $\widehat{G}$ defined on $[a,b]$ such that $\widehat{G}(b) = 0$ and $\left(\widehat{G}\right)' \in L^{r'}([a,b])$, where $r' = r/(r-1)$, we have*

$$(HK_r) \int_a^b f_n(x)\, \widehat{G}(x)\, dx \to (HK_r) \int_a^b f(x)\, \widehat{G}(x)\, dx$$

or, equivalently,

$$\int_a^b F_n(x) \left(\widehat{G}\right)'(x)\, dx \to \int_a^b F(x) \left(\widehat{G}\right)'(x)\, dx$$

where F_n and F are defined as in (2.2).

4 Distributional Derivatives

As we discussed in section 1, Bongiorno and Panchapagesan in 1995 described distributional derivatives on the space of Denjoy integrable functions. We follow the definitions given in [2].

Let H be the space of all Denjoy-integrable functions on $[a, b]$, endowed with the Alexiewicz norm. We denote by $\mathcal{H}$ the completion of H.

Let $\Omega = \{F \in \mathcal{C}[a, b] : F(a) = 0\}$. It is known that Ω is a Banach space with the sup norm. The space AC_0 of absolutely continuous functions F with $F(a) = 0$ is dense in Ω.

For each $h \in H$, let $\Phi_0(h)$ be the Denjoy primitive of h with $\Phi_0(h)(a) = 0$. Since $\Phi_0(h)$ is an ACG_* function on $[a, b]$ taking the value zero at a, and since $AC_0[a, b] \subseteq ACG_*[a, b] \subseteq \Omega$, Φ_0 is a linear isometry from H onto a dense subset of Ω.

A C^∞ function ϕ having support on $[a, b]$ is called a test function on $[a, b]$. A distribution is a bounded linear functional on the space of test functions. Given a continuous function F we denote its distributional derivative by D_F where for any test function

$$D_F(\phi) = -\int_a^b F\phi'.$$

When F is differentiable in the classical sense, we denote its derivative by F'.

We recall the following results.

Theorem 4.1. *[2] The following assertions hold:*

(i) $\mathbf{h} \in \mathcal{H}$ *if and only if* $\mathbf{h} = D_F$ *for some* $F \in \mathcal{C}[a, b]$.

(ii) For each $\mathbf{h} \in \mathcal{H}$ *there exists a unique* $F \in \Omega$ *such that* $\mathbf{h} = D_F$.

(iii) The mapping $\Phi : \mathcal{H} \to \Omega$ *given by* $\Phi(\mathbf{h}) = F$ *if* $D_F = \mathbf{h}$ *and* $F \in \Omega$ *is well defined and is an onto linear isometry extending* Φ_0. *Thus the unique isometric extension of* Φ_0 *to* $\mathcal{H}$ *is precisely the map* Φ *given above.*

We propose a characterization of the completion of HK_r in terms of distributional L^r derivatives on the space of HK_r integrable functions.

We denote by $\mathcal{HK_R}$ the completion of HK_r.

Given a function $F \in L^r$ we denote its distributional derivative by $D_{r,F}$ and, when F is differentiable, its derivative by F'.

Theorem 4.2. *[8]* $\mathbf{h} \in \mathcal{HK_R}$ *if and only if* $\mathbf{h} = D_{r,F}$ *for some* $F \in L^r$.

5 Future directions

We are planning to investigate relatively compact and relatively weakly compact subsets of HK_r and $\mathcal{HK}_{\mathcal{R}}$. Moreover, some applications to Fourier transforms will be studied.

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