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GENERALIZED INTEGRALS AND ORTHOGONAL SERIES WITH BANACH-SPACE-VALUED COEFFICIENTS

During the past decade it was a popular topic in vector-valued Fourier analysis to investigate if classical results about scalar-valued functions remain valid if the functions considered take values in some Banach space. The most desirable case is if the results remain true for any Banach space. The opposite case occurs if the extension is possible only for functions with values in finite dimensional spaces. The third case is that it depends on the structure of the Banach spaces considered whether a result can be extended to the vector-valued setting. A prominent example of the latter case is the vector-valued extension of Carleson's celebrated theorem on pointwise convergence of Fourier series which is possible only in the case of UMD (unconditionality of martingale differences) spaces and was obtained recently (see [1]) for a wide class of these spaces in the case of Fourier series with respect to Walsh and trigonometric systems. Other examples are given by the theory of type and cotype of Banach spaces.

We consider here the problem of recovering, by generalized Fourier formulae, the vector-valued coefficients of series with respect to some classical orthogonal systems. To solve this problem some generalizations of Bochner and Pettis integrals are introduced. In the case of Walsh and Haar series (see [2]) a suitable integral is a dyadic version of the Henstock or the Henstock-Pettis integrals. In the simplest case of convergence (strong or weak) everywhere these integrals solve the problem with coefficients from any Banach space.

We recall the definition of the *Haar system* on $[0,1]$. Let $\chi_0(x) \equiv 1$. If

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$n = 2^k + i - 1$, $k = 0, 1, \ldots$, $i = 1, 2, \ldots, 2^k$, we put

$$\chi_n(x) := \begin{cases} 2^{k/2}, & \text{if } x \in \left(\frac{2i-2}{2^{k+1}}, \frac{2i-1}{2^{k+1}}\right), \\ -2^{k/2}, & \text{if } x \in \left(\frac{2i-1}{2^{k+1}}, \frac{2i}{2^{k+1}}\right), \\ 0, & \text{if } x \in (0,1) \setminus \left[\frac{2i-2}{2^{k+1}}, \frac{2i}{2^{k+1}}\right]. \end{cases}$$

At each point of discontinuity $x \in (0,1)$ we put $\chi_n(x) = \frac{1}{2}(\chi_n(x+0) + \chi_n(x-0))$ and at $x = 0$ and $x = 1$ we define Haar functions to be continuous from the right and from the left, respectively.

The *Walsh functions* on $[0,1]$ are defined using dyadic expansions of natural numbers n and $x \in [0,1)$. Let $n = \sum_{j=0}^{\infty} \varepsilon_j 2^j$ with $\varepsilon_j = 0$ or 1, and $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ with $x_j = 0$ or 1, with the stipulation that for the dyadic rationals x we use only finite expansions. With this notation we put

$$w_n(x) = (-1)^{\sum_{j=0}^{\infty} \varepsilon_j x_j}$$

We denote by $\mathcal{I}$ the family of all *dyadic intervals*

$$I_j^{(n)} := \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right], \quad 0 \le j \le 2^n - 1, \quad n = 0, 1, 2, \ldots.$$

The *dyadic derivation basis* $\mathcal{B}$ is defined as the collection of sets

$$\beta_\delta := \{(I, x) : I \in \mathcal{I}, \quad x \in I \subset U(x, \delta(x))\}$$

where δ is the so-called *gauge*, i.e., a positive function defined on $[0,1]$, and $U(x,r)$ denotes the neighborhood of x of radius r. So we have

$$\mathcal{B} := \{\beta_\delta : \quad \delta : K \to (0, \infty)\}.$$

A *β_δ-partition* is a finite collection π of elements of β_δ, where the distinct elements (I', x') and (I'', x'') in π have I' and I'' non-overlapping, i.e., they have no inner points in common.

In this terms a dyadic integral of Henstock type is difined in the following way (for the real-valued case see [3]):

Definition. *Let X be a Banach space. A function $f : [0,1] \to X$ is said to be* Henstock integrable with respect to the dyadic basis $\mathcal{B}$ *or H_d-integrable on* $[0,1]$ *, with H_d-integral $A \in X$, if for every $\varepsilon > 0$, there exists a gauge δ such that for any β_δ-partition π of* $[0,1]$ *we have:*

$$|| \sum_{(I,x) \in \pi} f(x)|I| - A|| < \varepsilon.$$

We denote the integral value A by $(H_{\mathcal{B}}) \int_L f$.

Theorem 1. *If a Walsh series is strongly convergent to a sum f everywhere in $[0, 1]$ outside some countable set then f is H_d-integrable on $[0, 1]$ and the coefficients of the series are H_d-Fourier coefficients of f with respect to the Walsh system.*

Theorem 2. *If a Haar series is strongly convergent to a sum f everywhere in $[0, 1]$ then f is H_d-integrable on $[0, 1]$ and the coefficients of the series are H_d-Fourier coefficients of f with respect to the Haar system.*

Similar results hold for the case of weak convergence of the series. In this case a suitable integral is the *dyadic Henstock-Pettis integral* which is defined using the real-valued dyadic Henstock integral in the same way as usual Pettis integral is defined using the Lebesgue integral.

So the results on recovering the coefficients turns out to remain valid for *any* Banach space. At the same time some nice properties of Fourier series in the sense of these generalized integrals remain valid only for functions with values in finite dimensional spaces. A typical example:

Theorem 3. *For any infinite-dimensional Banach space there exists a function with values in this space such that its Fourier-Henstock-Haar series diverges almost everywhere.*

Moreover, the rate of growth of the partial sums in the above theorem can be $n^{\frac{1}{2}-\varepsilon}$ for any $\varepsilon > 0$. And this rate of divergence is close to describing the *worst* type of behavior of those partial sums that can occur in an *arbitrary* infinite-dimensional Banach space. In fact the growth $O(n^{\frac{1}{2}})$ for Fourier-Pattis-Haar series can be achieved for no Banach space with the Orlicz property, i.e., for spaces on which the identity operator is (2,1)-summing.

References

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