On some trigonometric polynomials with extremally small uniform norm

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For $f \in L^p(0, 2\pi)$ we set

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p\,dx\right)^{1/p} \qquad \text{for } 1 \le p < \infty,$$
$$\|f\|_\infty = \operatorname*{ess\,sup}_{[0,2\pi]} |f(x)| \qquad \text{for } p = \infty.$$

Let n be a natural number. Denote by E_n the space of real trigonometric polynomials of the form

$$t(x) = \sum_{k=2^{n-1}}^{2^n-1} a_k \cos kx + b_k \sin kx.$$

We prove the following result (see [1]).

Theorem. *Let $\varepsilon \in (0,1)$ be a real number, let $1 \leqslant k_1 < k_2 < \ldots < k_n < \ldots$ be a sequence of natural numbers, let L_n be a subspace of E_{k_n} such that $\dim L_n \geqslant \varepsilon \dim E_{k_n}$, $n = 1, 2, \ldots$. Then there exist trigonometric polynomials $t_n \in L_n$, $n = 1, 2, \ldots$, such that $\|t_n\|_\infty \leqslant 1$, $\|t_n\|_1 \geqslant c(\varepsilon)$, $n = 1, 2, \ldots$, and*

$$\Big\|\sum_{j=1}^n t_j\Big\|_\infty \leqslant \sqrt{n}, \quad n = 1, 2, \ldots, \tag{1}$$

where $c(\varepsilon) > 0$ is a constant which depends only on ε.

Note that if trigonometric polynomials $\{t_n(x)\}_{n=1}^\infty$ are orthogonal and $\|t_n\|_1 \geqslant \alpha > 0$, $n = 1, 2, \ldots$, then

$$\Big\|\sum_{j=1}^n t_j\Big\|_\infty \geqslant \frac{1}{\sqrt{2\pi}}\Big\|\sum_{j=1}^n t_j\Big\|_2 = \frac{1}{\sqrt{2\pi}}\left(\sum_{j=1}^n \|t_j\|_2^2\right)^{1/2} \geqslant$$

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{2\pi}\sum_{j=1}^n \|t_j\|_1^2\right)^{1/2} \geqslant \frac{\alpha}{2\pi}\sqrt{n},$$

$n = 1, 2, \ldots$. Therefore the order $\sqrt{n}$ on the right-hand side of the inequality (1) can not be decreased.

Bibliography

[1] A. O. Radomskii, *On nonequivalence of the* C- *and* QC-*norms in the space of trigonometric polynomials*, Sb. Math. **207** (2016), no. 12, 1729–1742.