A CHARACTERIZATION OF THE WEAK RADON-NIKODÝM PROPERTY BY FINITELY ADDITIVE INTERVAL FUNCTIONS

1 Preliminaries

Let \([0, 1]\) be the unit interval of the real line \(\mathbb{R}\) equipped with the usual topology and the Lebesgue measure \(\lambda\). We denote by \(I\) the family of all nontrivial closed subintervals of \([0, 1]\) and by \(L\) the family of all Lebesgue measurable subsets of \([0, 1]\).

Throughout this paper \(X\) is a Banach space. If \(\mu\) is an outer measure on \([0, 1]\), then by \(\mu \ll \lambda\) we mean that \(\lambda(E) = 0\) implies \(\mu(E) = 0\). A mapping \(\nu: L \to X\) is said to be an \(X\)-valued measure if \(\nu\) is countably additive in the norm topology of \(X\). \(\nu\) is said to be \(\lambda\)-continuous if \(|E| = 0\) implies \(\nu(E) = 0\). The variation of an \(X\)-valued measure \(\nu\) is denoted by \(|\nu|\). A function \(f: [0, 1] \to X\) is said to be scalarly measurable if for each \(x^* \in X^*\) the real function \(x^* f\) is measurable.

A Banach space \(X\) has the weak Radon-Nikodým property (see [4] or [5, Theorem 11.3]) if and only if for every measure \(\nu: L \to X\) of \(\sigma\)-finite variation, that is absolutely continuous with respect to the Lebesgue measure, there exists a Pettis integrable function \(f: [0, 1] \to X\) such that

\[
\nu(E) = \int_E f(t) \, dt, \quad \text{for every set } E \in L.
\]
More information on Pettis-integrable functions can be found in [5] and [6]. A partition in \([0, 1]\) is a finite collection of pairs \(P = \{(I_1, t_1), \ldots, (I_p, t_p)\}\), where \(I_1, \ldots, I_p\) are non-overlapping subintervals of \([0, 1]\) and \(t_i \in I_i, \ i = 1, \ldots, p\). Given a subset \(E\) of \([0, 1]\), we say that the partition \(P\) is anchored on \(E\) if \(t_i \in E\) for each \(i = 1, \ldots, p\). If \(\bigcup_{i=1}^p I_i = [0, 1]\) we say that \(P\) is a partition of \([0, 1]\). A gauge on \(E \subset [0, 1]\) is a positive function on \(E\). For a given gauge \(\delta\), we say that a partition \(\{(I_1, t_1), \ldots, (I_p, t_p)\}\) is \(\delta\)-fine if \(I_i \subset (t_i - \delta(x_i), t_i + \delta(x_i)), \ i = 1, \ldots, p\).

**Definition 1.** A function \(f : [0, 1] \to \mathbb{R}\) is said to be Henstock-Kurzweil integrable, (or \(HK\)-integrable), on \([0, 1]\), if there exists \(w \in \mathbb{R}\) with the following property: for every \(\epsilon > 0\) there exists a gauge \(\delta\) on \([0, 1]\) such that
\[
\left| \sum_{i=1}^p f(t_i)|I_i| - w \right| < \epsilon,
\]
for each \(\delta\)-fine partition \(P = \{(I_1, t_1), \ldots, (I_p, t_p)\}\) of \([0, 1]\).

We set \(w := (HK) \int_0^1 f d\lambda\).

It is known that if \(f : [0, 1] \to \mathbb{R}\) is \(HK\)-integrable on \([0, 1]\) and \(I \in \mathcal{I}\), then \(f\chi_I\) is also \(HK\)-integrable on \([0, 1]\). We say in such a case that \(f\) is \(HK\)-integrable on \(I\). We call the additive interval function \(F(I) := (HK) \int_I f d\lambda\) the \(HK\)-primitive of \(f\).

**Definition 2.** A function \(f : [0, 1] \to X\) is said to be scalarly Henstock-Kurzweil integrable if for each \(x^* \in X^*\) the function \(x^*f\) is Henstock-Kurzweil integrable. A scalarly Henstock-Kurzweil integrable function \(f\) is said to be Henstock-Kurzweil-Pettis integrable (or \(HKP\)-integrable) if for each \(I \in \mathcal{I}\) there exists \(w_I \in X\) such that
\[
\langle x^*, w_I \rangle = (HK) \int_I x^* f d\lambda, \text{ for every } x^* \in X^*.
\]

We call \(w_I\) the Henstock-Kurzweil-Pettis integral of \(f\) over \(I\) and we write \(w_I := (HKP) \int_I f d\lambda\). If \(I = [a, b]\), then we write \((HKP) \int_a^b f d\lambda\) instead of \((HKP) \int_{[a,b]} f d\lambda\). \(\square\)

We denote by \(HKP([0, 1], X)\) the set of all \(X\)-valued Henstock-Kurzweil-Pettis integrable functions on \([0, 1]\) (functions that are scalarly equivalent are identified). More information on \(HKP\)-integrable functions can be found in [3].

It is known that the \(HK\)-primitive (resp. \(HKP\)-primitive) \(F\) of a function \(f\) is continuous (resp. weakly continuous, i.e. \(x^*F\) is continuous for every \(x^* \in X^*\)).
Definition 3. Let $F : [0, 1] \to X$ be a function and $G$ be a non-empty subset of $[0, 1]$. If there is a function $F'_p : G \to X$ such that for each $x^* \in X$

$$\lim_{h \to 0} \frac{x^*F(t + h) - x^*F(t)}{h} = x^*(F'_p(t)),$$

for almost all $t \in G$, then $F$ is said to be pseudo-differentiable on $G$ (the exceptional sets depend on $x^*$), with a pseudo-derivative $F'_p$.

2 Variational Measures

Throughout the letter $\Phi$ will denote an arbitrary additive interval function $\Phi : I \to X$ that is identified with the point function $\Phi(t) = \Phi([0, t]), t \in [0, 1]$.

Definition 4. Given $\Phi : I \to X$, a gauge $\delta$ and a set $E \subset [0, 1]$ we define

$$\text{Var}(\Phi, \delta, E) := \sup \left\{ \sum_{i=1}^{p} ||\Phi(I_i)|| : \{(I_i, t_i) : i = 1, \ldots, p\} \delta\text{-fine partition anchored on } E \right\}.$$

Then we set

$$V_\Phi(E) := \inf \{ \text{Var}(\Phi, \delta, E) : \delta \text{ gauge on } E \}.$$

We call $V_\Phi$ the variational measure generated by $\Phi$. It is known that $V_\Phi$ is a metric outer measure on $[0, 1]$ (see [7]). In particular, $V_\Phi$ is a measure over all Borel sets of $[0, 1]$.

Definition 5. We say that the variational measure $V_\Phi$ is $\sigma$-finite if there is a sequence of (pairwise disjoint) sets $F_n$ covering $[0, 1]$ and such that $V_\Phi(F_n) < \infty$, for every $n \in \mathbb{N}$.

Thomson (see [7, Theorem 3.15]) proved that $V_\Phi$ has the so called measurable cover property, that is if $A \subset [0, 1]$, then there exists $B \in \mathcal{L}$ such that $B \supset A$ and $V_\Phi(B) = V_\Phi(A)$. It follows from this that the sets $F_n$ in the previous definition can be taken from $\mathcal{L}$.

Proposition 6. [1] If $V_\Phi \ll \lambda$, then $\Phi$ is continuous on $[0, 1]$ and $V_\Phi$ is $\sigma$-finite.

We recall that a function $\Phi : [0, 1] \to X$ is said to be $BV_*$ on a set $E \subseteq [0, 1]$ if $\sup \sum_{i=1}^{n} \omega(\Phi(J_i)) < +\infty$, where the supremum is taken over all finite collections $\{J_1, \ldots, J_n\}$ of non overlapping intervals from $I$ with end-points in $E$, and the symbol $\omega(\Phi(J))$ stands for $\sup\{||\Phi(u) - \Phi(z)|| : u, z \in J\}$. The function $\Phi$ is said to be $BVG_*$ on $[0, 1]$ if $[0, 1] = \bigcup_{n} E_n$ and $\Phi$ is $BV_*$ on each $E_n$. 
Proposition 7. [1] $V_\phi$ is $\sigma$-finite if and only if $\Phi$ is BVG$_*$ on $[0,1]$.

The following theorem is the main result:

Theorem 8. [1] Let $X$ be a Banach space. Then the following conditions are equivalent:

(i) $X$ has the weak Radon-Nikodym property;
(ii) If $\Phi : I \to X$ is $BV_*$ on $[0,1]$, then $\Phi$ is pseudo-differentiable on $[0,1]$;
(iii) If $\Phi : I \to X$ is BVG$_*$ on $[0,1]$, then $\Phi$ is pseudo-differentiable on $[0,1]$;
(iv) If $V_\phi$ is $\sigma$-finite, then $\Phi$ is pseudo-differentiable on $[0,1]$;
(v) If $V_\phi \ll \lambda$, then $\Phi$ is pseudo-differentiable on $[0,1]$;
(vi) If $V_\phi \ll \lambda$, then $\Phi$ is pseudo-differentiable on $[0,1]$, $\Phi' \in HKP([0,1],X)$ and

\[ \Phi(I) = \left( HKP \right) \int_I \Phi'_p \, d\lambda, \quad \text{for every } I \in I; \]

(vii) If $V_\phi \ll \lambda$, then there exists $f \in HKP([0,1],X)$ such that

\[ \Phi(I) = \left( HKP \right) \int_I f \, d\lambda, \quad \text{for every } I \in I. \]

References