Selected Ideas in the Mathematical Career of F. S. Cater

Dedicated to
Paul White
Whose teaching inspired my
interest in real variables

Preface

This pamphlet contains a collection of selected ideas from the mathematical career of F. S. Cater. In each case we give a reference of a paper or part of a paper. However, on occasion we do not have a reference so we write a short note (paper) in this pamphlet.

One purpose of the pamphlet is to gather these ideas together so we do not have to hunt them down in the literature. We can see the picture as a whole and identify quickly what we want to read. The best place to read the pamphlet is in a library where the Real Analysis Exchange and certain other journals are on the shelf, or by a computer that serves the same purpose.

Most of my career, and these ideas, are in real variables (real analysis) though there are occasional forays into linear algebra and topology (consult the short Chapter IV, for example).

The Chapters gather ideas that are most connected. In Chapter I we consider derivatives and differentiation, and nowhere differentiable functions. In Chapter II we consider families of continuous functions. For example, this may include $C(X)$ or some subfamily of $C(X)$ for a topological space X — in particular when X is a subset of R. In Chapter III we consider absolutely continuous functions and N-functions (f is an N-function if f maps sets of measure zero to sets of measure zero). Chapter IV is a short chapter (4 items) on the algebra of matrices. We have real variables again when the entries in the matrices are real numbers. In Chapter V we have alternative arguments. These are essentially different from the arguments they replace. For example, the alternative argument may be much simpler, or much shorter than the conventional argument. Another possibility is that the alternative argument requires much less sophisticated background than the conventional argument. Finally, Chapter VI is called Variety. It consists chiefly of ideas that we did not see fit to insert in other Chapters. In the first item of Chapter VI we find that my Erdös number is one.

Some items could possibly be in more than one Chapter. Many ideas in my papers were not included because I wanted to reduce the length. Then too, after many years there is much that may have interested me at one time, but no longer does. Sorry if I deleted your favorite topic in my papers. Finally, I selected items with the pamphlet as a whole in mind.

F. S. Cater
Portland, Oregon
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Contents

Chapter I

Derivatives and Differentiation

I.1 A typical nowhere differentiable function

After much experimenting over a period of years, we discovered that the continuous real valued function

$$F(x) = \sum_{n=1}^{\infty} 2^{-n!} \cos(2^{(2n)!}x)$$

satisfies the properties:

(i) At each x either $D^+F(x) = -D_-F(x) = \infty$ or $D^-F(x) = -D_+F(x) = \infty$, and the set of x where either equation does not hold is a first category set of measure zero,

(ii) At each x, $[D_-F(x), D^-F(x)] \cup [D_+F(x), D^+F(x)] = [-\infty, \infty]$,

(iii) Each of the four sets $\{x : F'_+(x) = \infty\}$, $\{x : F'_+(x) = -\infty\}$, $\{x : F'_-(x) = \infty\}$, $\{x : F'_-(x) = -\infty\}$ contains a perfect set in every interval, and hence has the power of the continuum in every interval.

It is known that in the metric space of continuous functions on $[0, 1]$ under the uniform norm, functions satisfying (i), (ii) and (iii) form a residual set. However it is difficult to find a succinct definition of such a function, like ours. Our paper is F.S. Cater "A typical nowhere differentiable function," *Canadian Math. Bull.* **26**(2), 1983, 149-151.

To read more about functions satisfying (i), (ii) and (iii) consult, for example, K.M. Garg "On a residual set of continuous functions," *Czechoslovak Math. Journal* **20** (1970), 537-543.

I.2 Nondifferentiability of a certain sum

Let $b > 1$ (b is not necessarily an integer) and let c_n be any real number for each index n. Let $K(x)$ denote the distance from x to the nearest integer. In connection with their own work in the 1990s, A. Baouche and S. Dubuc inquired about the differential status of the function

$$F(x) = \sum_{n=0}^{\infty} K(b^n x + c_n)/b^n, \qquad (b > 1).$$

In F.S. Cater, "Remark on a function without unilateral derivatives," *Journal of Math. Analysis and Applications* **182** (3), March 1994, 718-721, we gave a partial answer.

We proved that F has no finite unilateral derivative at any point if $b \geq 10$. At this writing, I do not know what the status of F is for $1 < b < 10$.

I.3 A derivative often zero and discontinuous

We give a constructive definition of a derivative h on $[0,1]$ that is discontinuous almost everywhere on $[0,1]$ but vanishes on a set of positive measure in each subinterval of $[0,1]$. Our paper is F.S. Cater "A derivative often zero and discontinuous," *Real Analysis Exchange* **11**, (1985-86), 265-270.

It was written in reply to Clifford Weil "The space of bounded derivatives," *Real Analysis Exchange* **3**, (1977-78), 38-41, where a category argument was used to prove the existence of a derivative on $[0,1]$ that vanishes on a dense subset of $[0,1]$ but is nonzero almost everywhere.

I.4 An increasing continuous function with Dini derivates 0 and ∞

In F.S. Cater "On the Dini derivates of a particular function," *Real Analysis Exchange* **25** (1), 1999-2000, 1-4, we constructed a continuous strictly increasing function f such that at each point x, either $D_+f(x) = 0$ or $D^+f(x) = \infty$, and at each point x, either $D_-f(x) = 0$ or $D^-f(x) = \infty$. Moreover, 0 or ∞ is a derivate (left or right) at each point.

I.5 Constructing nondifferentiable functions from concave functions

Let (a_n) be a sequence of nonnegative real numbers such that $\sum_n^\infty a_n < \infty$. Let (b_n) be a strictly increasing sequence of positive numbers such that b_n divides b_{n+1} for each n, and $(a_n b_n)$ does not converge to zero.

Let f be a continuous function mapping the real line onto the interval $[0, 1]$ such that $f(1) = 1$, $f(0) = f(2) = 0$, and f is concave down on the interval $[0, 2]$. Let $f(x + 2) = f(x)$ for each x.

In F.S. Cater, "Constructing nowhere differentiable functions from convex functions," *Real Analysis Exchange* **28** (2), 2002-2003, 617-622, we proved that

$$\sum_{j=1}^{\infty} a_j f(b_j x)$$

has a finite left or right derivative at no point.

In this way, we can construct nowhere differentiable functions out of concave (and convex functions. Several examples were offered.

Later it was pointed out to me that I had inadvertently interchanged the definitions of "convex" and "concave"in the paper. Sorry about that. I always thought that a bump in the road  was convex, but a pot hole in the road  was concave. No matter.

I.6 Sums of jump functions

By a jump function centered at a point x in R we mean a real functions on R, constant on the intervals $(-\infty, x)$ and (x, ∞) such that

$$f(x-) \leq f(x) \leq f(x+) \text{ and } f(x-) < f(x+).$$

In G. Piranian "The derivative of a monotone discontinuous function," *Proc. of the Amer. Math. Soc*, **16**(2), 1965, 243-244, George Piranian proved that if S is a countable G_δ-set in R, then there exists a nondecreasing function f on R with infinite derivative at each point in S and zero derivative at each point not in S. He made f the sum of jump functions centered at the points in S. Thus f was also discontinuous at each point of S in his construction.

In F.S. Cater "On functions differentiable on complements of countable sets," *Real Analysis Exchange*, **32**(2), 2006-2007, 527-536, we proved that if g

is a nondecreasing bounded function with zero derivative at all but countable many points, and g has infinite derivative at every other point, then g can be expressed as the sum of (countably many) jump functions. Furthermore, the set of all points where g has infinite derivative is necessarily a nowhere dense G_δ-set.

For example, there exists no nondecreasing function g, discontinuous at every rational point, that has zero derivative at every irrational point.

I.7 On the range of a real function

Let f be a real function on R, and let $\{I_v\}$ denote a family of intervals covering a set S such that $m(S \cap I_v) \geq m(f(S \cap I_v))$ for each I_v. (Here m denotes Lebesgue outer measure.) In F.S. Cater, "Note on the outer measures of images of sets," *Real Analysis Exchange*, **26**(2), 2000-2001, 827-830, we proved that $m(f(S)) \leq 2m(S)$. We showed by example that no coefficient less than 2 will suffice here in general.

Observe that this does not necessarily involve derivatives.

I.8 Using derivates to partition R

In this item we use derivates of a function to give a constructive definition of a partition of R into continuum many $F_{\sigma\delta}$-sets, each of which meets every subinterval of R in continuum many points. We do not have a paper on this construction so we write it here. We do not use or need the Continuum Hypothesis.

Let f be a continuous nondecreasing functions from $[0,1]$ onto $[0,1]$ that has zero derivative almost everywhere. For each n positive, negative or zero and $x \in [0,1]$, put

$$f_0(n+x) = n + f(x).$$

Then f_0 is a continuous nondecreasing function from R onto R with zero derivative almost everywhere. Put

$$H(x) = \sum_{n=0}^{\infty} 2^{-n} f_0(nx).$$

By the "other" Fubini Theorem (consult (17.18) of Hewitt & Stromberg, *Real and Abstract Analysis*, Springer, New York, 1965), $H' = 0$ almost everywhere. Moreover H is strictly increasing on R.

For each extended real number r, $0 \leq r \leq \infty$, put

$$T_r = \{x : D^+H(x) = r\}.$$

The T_r apparently form a partition of R composed of continuum many sets. It remains to prove that the desired hypotheses are satisfied.

Lemma 1. For each extended real number r, $0 < r < \infty$, T_r meets every interval in continuum many points.

Proof. The complement of T_0 has measure zero, so the set

$$H\{x : 0 < D^+H(x) < \infty\}$$

has measure zero. Because D^+H vanishes on T_0 it follows that the set

$$H\{x : D^+H(x) = 0\}$$

also has measure zero. We deduce that for any interval I,

$$H\{x \in I : D^+H(x) = \infty\}$$

has positive measure. It follows that $I \cap T_\infty$ has the power of the continuum. But the complement of T_0 has measure zero, so $I \cap T_0$ also has the power of the continuum.

So now let r be a real number $0 < r < \infty$, and let I be any interval. Let g be a continuous function on R such that $g(x) = H(x)$ for $x \in I$, and $g'(x) > r$ for x outside the closure of I.

It follows that $D^+g > r$ on a dense subset of R. There is evidently a point $x_0 \in I$ where

$$D^+g(x_0) = D^+H(x_0) = 0 < r$$

because T_0 is dense in R. It follows from Anthony Morse "Dini derivates of a continuous function", *Proc. Amer. Math. Soc*, **5** (1954), 126-130, that $\{x : D^+g(x) = r\}$ has continuum many points. It follows that

$$I \cap T_r = \{x \in I : D^+H(x) = r\}$$

has continuum many points.

To complete the argument, we need one more well-known Lemma. We include a proof of it for any one who wants one written here.

Lemma 2. For each extended real number r, $0 < r < \infty$, the set T_r is an $F_{\sigma\delta}$-subset of R.

Proof. Let r be a positive real number and let n be a positive integer. Then it follows that the set

$$S_n = \{x : \text{ there exists a } y > x \text{ such that } (H(y) - H(x)) > r(y - x),\ 0 < y - x < n^{-1}\}$$

is open. Then the set $P_r = \cap_{n=1}^{\infty} S_n$ is a G_δ-set. Clearly P_r contains the set $\{x : D^+H(x) > r\}$ and P_r is disjoint from the set $\{x : D^+H(x) < r\}$. It follows that

$$T_\infty = \bigcap_{k=1}^{\infty} P_k$$

is a G_δ-set and consequently T_∞ is also an $F_{\sigma\delta}$-set. Furthermore

$$T_r = \left[\bigcap_{k=1}^{\infty} P_{r-k^{-1}}\right] \setminus \left[\bigcup_{k=1}^{\infty} P_{r+k^{-1}}\right]$$

is a G_δ-set minus a $G_{\delta\sigma}$-set, and in turn is the intersection of a G_δ-set with an $F_{\sigma\delta}$-set. Finally, T_r is the intersection of two $F_{\sigma\delta}$-sets and is an $F_{\sigma\delta}$-set.

Observe that the function f completely determines the partition in this argument. Consequently, when f is Lebesgue's singular function, the definition of f is constructive and the definition of the partition is likewise constructive.

You also may be interested in the paper *Amer. Math. Monthly* **91** (9), November 1984, 564-566, in which we partitioned $(0, 1)$ into countably many measurable sets that each meet every subinterval of $(0, 1)$ in a second category set of positive measure. However, we cannot make these measurable sets all Borel sets.

Chapter II

Families of continuous functions

II.1 Function lattices.

Let U be a locally compact Hausdorff space that is not compact. Let $L(U)$ denote the family of all continuous real valued functions f on U such that for some nonzero number p, depending on f, $f-p$ vanishes at infinity. Now let S be a locally compact Hausdorff space. Define $T(S)$ to be $C(S)$ if S is compact, and define $T(S)$ to be $L(S)$ if S is not compact. In F.S. Cater "Some lattices of continuous functions on locally compact spaces", *Real Analysis Exchange* **33**(2), 2007-2008, 285-290, we proved that for any locally compact spaces S_1 and S_2, S_1 and S_2 are homeomorphic spaces if and only if $T(S_1)$ and $T(S_2)$ are isomorphic lattices.

Thus $T(S)$ assumes a similar role for locally compact spaces S that $C(X)$ assumes for compact spaces X.

II.2 Nonlinear mappings

Let X be a compact Hausdorff space, and let D(X) denote the family of all continuous functions f on X satisfying $0 \leq f \leq 1$. In S. Cater "A nonlinear generalization of a theorem on function algebras," **Amer. Math. Monthly 74** (#4), June-July, 1967, 682-685, we proved the following result.

Theorem 1. Let X be a compact Hausdorff space and let u be a mapping of $D(X)$ into the unit interval $[0, 1]$ such that
(1) $u(fg) = u(f)u(g)$ for $f, g \in D(X)$,
(2) $u(1-f) = 1-u(f)$ for $f \in D(X)$.

Then there is a unique point $x_0 \in X$ such that $u(f) = f(x_0)$ for all $f \in D(X)$.

This result will be used in the next Item.

(Note: At that time I was known as S. Cater, an exaggerated attempt at brevity on my part. Now there exists another S. Cater, about whom I know nothing.)

II.3 Function rings

For a compact Hausdorffspace X let $C(X)$ denote the ring of continuous functions on X. For a locally compact, noncompact space Y let $G(Y)$ denote the ring of continuous functions f on Y such that there exists an integer p, depending on f, for which $f - p$ vanishes at infinity on Y. For a locally compact space W, let $P(W) = C(W)$ if W is compact, and let $P(W) = G(W)$ if W is not compact. In Frank S. Cater "Variations on a theorem on rings of continuous functions," *Real Analysis Exchange* **24** (2), 1998-1999, 579-588, we proved that locally compact spaces W_1 and W_2 are homeomorphic spaces if and only if $P(W_1)$ and $P(W_2)$ are isomorphic rings.

Thus the ring $P(W)$ plays a similar role for locally compact W that the ring $C(X)$ plays for compact X.

Finally, for real numbers a and b put $a * b = ab - a$. All this works just as well when the ring isomorphisms are replaced by bijections ϕ preserving the one operation $*$:

$$\phi(f(x) * g(x)) = \phi(f(x)) * \phi(g(x)).$$

One typo: on page 583, "homomorphism" should be "homeomorphism."

(This time they called me "Frank S. Cater" which is the name I use in business.)

II.4 Spaces of functions

In F.S.Cater "On sparse subspaces of $C[0, 1]$," *Real Analysis Exchange* **31** (1), 2005-2006, 7-12, we proved that there exists a subspace H of $C[0, 1]$ under the uniform metric that is homeomorphic to the full space $C[0, 1]$, even though H consists only of infinitely many times differentiable members of $C[0, 1]$. Likewise, there is a subspace H_1 of $C[0, 1]$, composed only of singular functions of bounded variation, such that H_1 is homeomorphic to $C[0,1]$. Furthermore, there exists a subspace H_2 of $C[0,1]$, composed only of nowhere differentiable functions, such that H_2 is homeomorphic to $C[0,1]$.

II.5 Lattice automorphisms

Let ϕ be a lattice automorphism of the lattice $C(X)$ where X is a compact Hausdorff space. Thus for f, $g \in C(X)$, $f-g \geq 0$ if and only if $\phi(f)-\phi(g) \geq 0$. We say that ϕ is increasing provided for f, $g \in C(X)$, $f-g$ never vanishes on X if and only if $\phi(f) - \phi(g)$ never vanishes on X. There are compact Hausdorff spaces X that admit lattice automorphisms that are not increasing. One such space is the Stone-Čech compactification of the real line.

In Theorem IV of F.S. Cater "Remark on a result of Kaplansky concerning $C(X)$," **Michigan Math. Journal12**(1965),97-103, we proved that every lattice automorphism must be increasing provided X is either locally connected or sequentially compact. I was surprised by the role of "locally connected" but I was not surprised by the role of "sequentially compact."

We deduce that the Stone-Čech compactification of the real line is connected but not locally connected.

(Note: After publication it was pointed out to me that there was a problem with Theorem II and its proof in the paper cited above. In retrospect, I should have written a short note on Theorem IV instead of the Michigan paper. Sorry. Theorem IV and its proof depend in no way on Theorem II.)

II.6 Functions with frequently infinite derivatives

In F.S. Cater, "On infinite unilateral derivatives," *Real Analysis Exchange* **33**(2), 2007-2008, 309-316, we proved that for any continuous function f on $[a,b]$, there exists a continuous function K on $[a,b]$ such that $K-f$ has zero derivative almost everywhere, and every subinterval I contains continuum many points where $K'_+ = \infty$, continuum many points where $K'_- = \infty$, continuum many points where $K'_+ = -\infty$, and continuum many points where $K'_- = -\infty$. Note that K and f have the same Dini derivates at almost every point. For certain functions f, for example, N-functions and functions of bounded variation, K can be selected so that the infinite derivatives are bilateral.

For more about N-functions, consult Chapter III.

II.7 On derivatives of functions of bounded variation

This Item is a supplement to our paper F.S. Cater "On the derivatives of functions of bounded variation," *Real Analysis Exchange* **26**(2), 2000-2001, 923-932.

Let F denote the family of all continuous functions of bounded variation on the interval $[0,1]$. The uniform metric is not complete on F. A better choice is the complete metric w for which

$$w(f,g) = |f(0) - g(0)| + \text{ total variation of } f - g \text{ on } [0,1].$$

We say that something is true of a "typical" function in F if the set of all functions in F for which it is not true is a first category subset of F.

For consistency, theorem III in our paper should read:
"The restriction of the derivative of a typical function f to its set of points of differentiability is unbounded in every subinterval". Indeed we proved that for typical functions f, the range of f' on any subinterval is dense in R.

To show that the absolute value bars can be removed in theorem I of our paper, argue as follows.

Proof. Let $f \in F$, $\epsilon > 0$, and let (a,b) be any subinterval of $[0,1]$.

We use Lemma 2 of our paper F.S. Cater "On infinite unilateral derivatives", *Real Analysis Exchange* **33**(2), 2007/2008, pp. 309-216, to construct a continuous nondecreasing singular function $g \in F$ of total variation 2ϵ, and vanishing at 0, such that for any $h \in F$ with total variation less than ϵ, $(g+h)'(x) = \infty$ at continuum many points x of (a,b) at which f is differentiable. Then $(f+g+h)'(x) = \infty$ at all such points x. Let $K_{a,b}$ denote the family of all functions $k \in F$ for which $k' = \infty$ at continuum many points in (a,b). It follows that $f+g$ is an interior point of $K_{a,b}$. Now $w(f+g,f) = 2\epsilon$, so $K_{a,b}$ contains an open dense subset of F because ϵ and f were arbitrary.

Let a, b run over the rational numbers in $(0,1)$. Then

$$K = \bigcap\nolimits_{a,b} K_{a,b}$$

has a first category complement in F. But any $p \in K$ satisfies $p' = \infty$ at continuum many points in any subinterval of $[0, 1]$. The corresponding result holds for $-\infty$. $\square$

We continue as follows. Fix an index $n > 0$, and typical $f \in F$. For any x satisfying $f'(x) = \infty$ select an interval (c, d) such that $f(d) - f(c) > d - c$, $x \in (c, d)$ and $d - c < n^{-1}$. Let S_n denote the union of all such intervals as x runs over all points where $f' = \infty$ and $n > 0$. Let T_n be defined in the same way where $f(d) - f(c) < -(d - c)$ and $f'(x) = -\infty$. It follows that

$$\bigcap_n (S_n \cap T_n)$$

is a residual subset of $[0, 1]$ and f can have no derivative, finite or infinite, at any point of this set. Finally, a residual subset of $[0, 1]$ must meet each subinterval of $[0, 1]$ in continuum many points.

We recapitulate:

For a typical function f in F and any subinterval I of $[0, 1]$, $f' = \infty$ at continuum many points of I, $f' = -\infty$ at continuum many points of I, and there are continuum many points of I where f has no derivative, finite or infinite. Of course every $f \in F$ is differentiable almost everywhere.

In the proof of theorem I in our paper we seemed to need the Continuum Hypothesis. It is not needed here.

You also might be interested in our paper "Most monotone functions are not singular," Ămerican Math. Monthly **89**(7), August-September, 1982, pp. 466-469.

Chapter III

Absolute continuity and the Lusin Property (N)

III.1 An equation for absolute continuity

Let f be a continuous function of bounded variation on $[a, b]$ and let L denote the length of $f[a, b]$. For each $x \in [a, b]$ let $k(x) = 0$ if the set

$$\{t \in [a, b] : f(t) = f(x)\}$$

is infinite, and $k(x) = 1/N$ if this set has N elements.

In Theorem 16 of F.S. Cater "On change of variables in integration," *Eötvös* (vols. XXII-XXIII), 1979-1980, 11-22, we proved that f is absolutely continuous on $[a, b]$ if and only if

$$\int_a^b k(x)|f'(x)|dx = L.$$

Thus we express the absolute continuity of a continuous function f of bounded variation in terms of an equation involving f'.

A better known necessary and sufficient condition is that f be an N-function on $[a, b]$, that is, f, maps sets of measure zero to sets of measure zero. This is called the Banach-Zarecki Theorem.

III.2 Summable and absolutely continuous functions

The Banach-Zarecki Theorem states that a necessary and sufficient condition for a continuous function f of bounded variation to be absolutely continuous on an interval is that f be an N-function on that interval.

In F.S. Cater "Some variations on the Banach-Zarecki Theorem," *Real Analysis Exchange*, **32**(2),547-552, we proved:

Corollary 1. Let f be a continuous N-function differentiable almost everywhere on $[a, b]$. Then f is absolutely continuous on $[a, b]$ if and only if there exists a summable function g on $[a, b]$ such that $g \geq f'$ almost everywhere on $[a, b]$.

Corollary 1 is a tolerable variation of (6.9), Chapter IX in S. Saks, *Theory of the Integral*, Second revised edition, Dover, New York, 1964. We hope it is of use and of interest in University teaching.

III.3 Knot points and N-functions

By a knot point p of a function f, we mean a point p at which the two upper Dini derivates of f are ∞, and the two lower Dini derivates of f are $-\infty$. In F.S. Cater "On continuous N-functions and an example of Mazurkiewicz", *Real Analysis Exchange*, **30**(1), 2004-2005, 201-206, we proved:

Corollary 3. Let f be a continuous function that is not an N-function on $[a, b]$, let K be the set of all knot points of f, and let $f(K)$ have measure zero. Then for any everywhere differentiable function g on $[a, b]$, $f + g$ is not an N-function on $[a, b]$.

As an example, let F be a monotone nonconstant continuous function on $[0, 1]$ with $F' = 0$ almost everywhere. Then for any everywhere differentiable function g on $[0, 1]$, $F + g$ is not an N=function on $[0, 1]$.

Mazurkiewicz constructed a continuous function $M(x)$ such that $M(x)+x$ is an N-function but $M(x)$ is not. Then $M(K)$ does not have measure zero.

For more about knot points, consult the work of K.M. Garg.

In this item we discussed a nexus between knot points and Masurkiewicz-type functions.

III.4 Completing an N-function example

In the *Real Analysis Exchange*, Summer Symposium 2002, p. 411, we posed the following research question.

Question. Given a nonconstant continuous N-function f, is there a continuous N-function g, depending on f, such that the sum $f + g$ is not an N-function.

The question was answered in the affirmative in Dusan Pokorny "On Lusin's (N)-property of the sum of two functions," *Real Analysis Exchange*, **33**(1), 2007-2008, 23-28.

So Pokorny's Theorem generalizes the Masurkiewicz example from linear functions to continuous functions.

We had a hand in this because we posed the research problem in print.

III.5 Compact sets and N-functions

Put $f(x) = x$ in Lemma 1 of F.S. Cater "On continuous N-functions and an example of Mazurkiewicz," *Real Analysis Exchange*, **30**(1), 2004-2005, 201-206, and obtain

Lemma 1. Let h be a continuous function on $[a, b]$ and let $S \subset [a, b]$ be a set of measure zero such that $h(S)$ does not have measure zero. Then there exists a compact subset T of S closure such that T has measure zero but $h(T)$ has positive measure.

So for a continuous function h to be an N-function it suffices that h maps compact sets of measure zero to sets of measure zero.

We will use this Lemma in the next Item.

III.6 N-functions relative to a closed set

Say that a function f is an N-function relative to the set P if f maps any subset of P of measure zero to a set of measure zero.

Here we pose the following question: For any uncountable closed set P of measure zero, do there exist continuous functions F and G on R, depending on P, that are N-functions relative to R such that the sum $F + G$ is not an N-function relative to P? We will answer this in the affirmative. But we did not publish it elsewhere, so we write it here.

We start with two continuous N-functions f and g with respect to R such that $f + g$ is not an N-function with respect to R. It follows from Item III(5) that there is an uncountable compact set X of measure zero such that $(f + g)(X)$ has positive measure. It follows that there is a perfect subset X_1 of X such that $X \backslash X_1$ is at most a countable set. Thus $(f + g)(X_1)$ has positive measure.

Now P has a closed bounded subset P_0 that is also uncountable. There is a compact perfect subset P_1 of P_0 such that $P_0 \backslash P_1$ is at most a countable set. Between any two complementary intervals of P_1 there are other complementary intervals of P_1. Likewise, between any two complementary intervals of X_1 there are other complementary intervals of X_1.

It follows that there is an order preserving bijection K of the set of all complementary intervals of P_1 onto the set of all complementary intervals of X_1. In an obvious manner K gives rise to an increasing function k of the complement of P_1 onto the complement of X_1 such that k is linear on each component of the complement of P_1.

In the natural way we extend k to an increasing homeomorphism of R onto R by assigning to each $x \in P_1$ the unique point $k(x)$ in X_1 that makes k everywhere increasing on R. Note that k then, is an N-function with respect to each component of the complement of P_1, and an N-function with respect P_1 because X_1 has measure zero. It follows that k is an N-functions with respect to R. Define the composite functions on R

$$F = f * k, \qquad G = g * k.$$

Now F and G are N-functions with respect to R because f, g and k are. But

$$F + G = (f + g) * k$$

maps the set P_1 of measure zero to the set $(f + g)(X_1)$ that has positive measure. It follows that $F + G$ is not an N-function with respect to P_1 or P.

REMARK. It can be shown (Item VI(12)) that any closed uncountable set contains an uncountable closed subset of measure zero. Thus it suffices in our argument that P be any uncountable closed set. I do not know what, if anything, can be concluded if P is only an uncountable set.

Chapter IV

Matrix algebra

All matrices are square, n by n, and the entries are in a commutative field, F.

IV.1 The matrix factor theorem

In Lemma 8 of F.S. Cater "Products of central collineations," *Linear algebra and its applications* **19** (1978), 251-274, we have a result I call the Matrix factor theorem. It can be described as follows:

Let M be a nonsingular n by n nonscalar matrix (not the product of a scalar with the identity matrix I). Let $x_1, x_2, ...x_n$ be scalars such that $x_1 x_2 ..., x_n = \det M$. Then there exist matrices $M_1, M_2, ..., M_n$ such that

(1) the product $M_1 M_2 ... M_n$ is similar to the matrix M,
(2) $\det M_j = x_j$ for $j = 1, 2, ..., n$,
(3) for each index $j = 1, 2, ..., n$, all the nonzero entries of $M_j - I$ lie in the $j - th$ column of $M_j - I$.

The developments of Lemmas 4 and 6 should be clearer so we will give other proofs here.

Let M be a nonsingular nonscalar matrix. Our arguments will be in a sequence of steps.

Step 1. M is similar to a nondiagonal matrix.

Proof. Let a_{ij} denote the i-th row, j-th column entry of M. Let M be diagonal. There is an index i with $a_{ii} \neq a_{11}$ because M is not a scalar matrix. Then M is similar to the matrix found by adding the i-th row of M to the first row of M, then subtracting the first column from the i-th column. The pattern is suggested by

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} M \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}.$$

Step 2. M is similar to a matrix whose 1-st row is not a scalar multiple of $(1, 0, ..., 0)$.

Proof. In view of Step 1, we assume that M is not a diagonal. Say $a_{ij} \neq 0$ where $i \neq j$. Then M is similar to the matrix found by interchanging the i-th and j-th rows of M, then interchanging the i-th and j-th columns. The result is a matrix with an entry equal to the value a_{ij} in the first row but not in the first column. The pattern is suggested by

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} M \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Step 3. Let p be any nonzero scalar in F. Then M is similar to a matrix whose first row is $(p, p, 0, ..., 0)$.

Proof. In view of Step 2, we can assume that for some index $j > 1$, $a_{1j} \neq 0$. We multiply the first row of M by $p/a_1 j$ and then multiply the first column by a_{1j}/p, to find that M is similar to a matrix with p in the first row and j-th column. The pattern is suggested by

$$\begin{pmatrix}p/a_{lj} & 0\\ 0 & 1\end{pmatrix} M \begin{pmatrix}a_{lj}/p & 0\\ 0 & 1\end{pmatrix}.$$

We add $(p - a_{11})/p$ times the j-th column to the first column and then add $(a_{11} - p)/p$ times the first row to the j-th row to find that M is similar to a matrix with p in the first row first column entry and p in the first row j-th column entry. If $j > 2$, we use the same procedure with the second column in place of the first column to find that M is similar to a matrix with p in the first row, first and second column entries. We proceed with the second column in place of the j-th column to convert all the first row entries after the second to zero. So M is similar to a matrix whose first row is $(p, p, 0, ..., 0)$. This is essentially Lemma 6 in our paper.

Step 4. Let $n = 2$. Let $x_1,\ x_2 \in F$ such that $x_1 x_2 = \det M$. Then M is similar to a matrix product of the form

$$\begin{pmatrix}x_1 & 0\\ \surd & 1\end{pmatrix} \begin{pmatrix}1 & \surd\\ 0 & x_2\end{pmatrix}.$$

Proof. By Step 6, we can assume that

$$M = \begin{pmatrix} x_1 & x_1 \\ a & b \end{pmatrix}.$$

Then

$$M = \begin{pmatrix} x_1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & b-a \end{pmatrix}.$$

Clearly $b - a = x_2$ because $x_1 x_2 = \det M$. This is essentially Lemma 4 in our paper.

Take Lemmas 5, 7, 8 and 9 as they appear in our paper. Lemma 8, then, is essentially the matrix factor theorem. We used Lemma 9 to give an answer to a question posed by Radjavi.

Here are two typos in our paper. On page 256, the matrix I_{n-1} should be I_{n-2}. On page 254, the first matrix appearing after the word "proof" can be better written. But this is obviated by the arguments here.

Here are some corollaries not included in our paper.

Corollary 1. Let A and B be nonsingular, nonscalar matrices that are not similar, but such that $\det A = \det B$. Let $|F| > 3$. Then A and B can be factored

$$A_1 A_2 ... A_n = A, \quad B_1 B_2 ... B_n = B$$

such that A_j is similar to B_j for $j = 1, 2, ..., n$.

To see this, observe that for $u \neq 0$, $u \neq 1$, diagonal $(u, 1, ..., 1)$ is similar to any matrix C with $\det C = u$ provided all the nonzero entries in $C - I$ lie in one column.

Corollary 2. Let M be a nonsingular matrix and let $c \in F$ such that $c \neq 1$ and $c^n = \det M$. Then there exist mutually similar matrices $M_1, M_2, ..., M_n$ such that $M = M_1 M_2 ... M_n$.

We leave the proof.

Corollary 3. Let M be a nonsingular matrix and let F be algebraically closed. Then there is a scalar s such that $sM = M_1 M_2 ... M_n$ where each ${M_j}^2$ is the identity matrix.

We leave the proof.

IV.2 Determinants and scalar mappings

Let p be an mapping from F to F. Let ϕ be the mapping from the set of n by n matrices into F as follows: For matrix A,

$$\phi(A) = p(\det A).$$

Observe the $\phi(AB) = \phi(BA)$ because

$$\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA).$$

In S. Cater, "Scalar valued mappings of square matrices," *Amer. Math. Monthly* **70**(2), 1963, pp. 163-169, we proved that the following are equivalent for a mapping ϕ from the set of n by n matrices into F

(1) $\phi(ABC) = \phi(CBA)$ for any matrices A, B, C,

(2) there exists a mapping p from F to F such that for all matrices A,

$$\phi(A) = p(\det A).$$

Moreover, ϕ is multiplicative if and only if p is multiplicative.

This paper was given an honor of sorts. It was reprinted in the Raymond W. Brink selected mathematical papers, volume 3, Algebra, pp. 321-327, published by the Mathematical Association of America, 1977.

IV.3 On multiplicative mappings

IV(3) was a precursor to IV(2). This time ϕ_1 and ϕ_2 are each multiplicative mappings from the set of n by n matrices to F, the field of complex numbers, that preserves conjugation:

$$\phi_i(AB) = \phi_i(A)\phi_i(B), \phi_i(A)^- = \phi_i(A^-),$$

for all A and B. Let $\phi_1(e^{1+i}I) = \phi_2(e^{1+i}I)$, In S. Cater, "On multiplicative mappings of operators", *Proc. of the Amer. Math. Soc.* **13**(1), 1962, pp. 55-58, we proved that $\phi_1 = \phi_2$ under these hypotheses. Moreover we proved that $\phi_1(A) = \det A$ if $\phi_1(e^{1+i}I) = e^{n+ni}$.

IV.4 Text Book

You may be interested in our linear algebra text book Frank S. Cater, Lectures on real and complex vector spaces, W. B. Saunders Company, Philadelphia, 1966.

It went out of print about 1971, but it still can be found in many university libraries. It sold few copies – probably fewer than a thousand – but it did receive good reviews. I wonder if it would have sold better if we had not written the functions on the right and the variables on the left. This notation was all the rage at that time. Note that in the reference in IV(2), the same notation was used.

Fortunately this notation is not so popular today. My feeling is that it causes confusion unless it is used in all branches of mathematics, not just linear algebra. In retrospect, I regret using that notation. I know it caused me confusion in teaching.

Chapter V

Alternative arguments

In this chapter we construct proofs of well-known results that are different from the usual proofs given. We try to make our arguments easier (simpler) than the accepted arguments and/or employ more elementary means (first principles). We hope to provide some worthwhile insights into these results as well.

V.1 On infinite unilateral derivatives

One result of S. Saks is that in the complete metric space $C[0,1]$ under the sup norm, the set of functions that have right (left) derivative $\infty(-\infty)$ at continuum many points in [0,1] form a residual subset of $C[0,1]$. Shorter proofs than his had been given, but they required sophisticated results. In F. S. Cater "An elementary proof of a theorem on unilateral derivatives," *Canadian Math. Bull.* **29**(3), 1986, pp. 341-343, we provided a relatively short argument using first principles.

We used the same technique to give a short proof that the functions in $C[0,1]$ that have continuum many knot points form a residual subset of $C[0,1]$.

V.2 A theorem of de la Vallée Poussin

We give a reasonable short proof from first principles of a classical theorem attributed to de la Vallée Poussin.

Theorem. Let f be a real valued function of bounded variation on the interval $[a, b]$ and let $V(x)$ denote the total variation of f on $[a, x]$. Let m denote Lebesgue outer measure. Then there exists a set $N \subset [a, b]$ such that

$$m(V(N)) = m(f(N)) = m(N) = 0$$

such that for any $x \in [a, b]\backslash N$, $f'(x)$ and $V'(x)$ exist, finite or infinite, and furthermore $V'(x) = |f'(x)|$.

Note that $m(N) = 0$ is easy to acquire from school analysis, but

$$m(N) = m(f(N)) = m(V(N)) = 0$$

is much harder to acquire. Our paper is F. S. Cater "A new elementary proof of a theorem of de la Vallée Poussin," *Real Analysis Exchange* **27**(1), 2001/2002, pp. 393-396.

We hope our argument will make this classical Theorem more accessible in the Universities.

V.3 Geodesics on spheres in Hilbert space

Here we make a foray into geometry. Let S be a sphere in a Euclidean space of dimension greater then 2, or a real Hilbert space, and let A and B be two points on S that are not antipodal. We give an elementary geometry proof that the shortest path on S joining points A and B lies on the great circle joining A and B (that is, the intersection of S with the plane through A, B and the center of S.)

We consider all the continuous rectifiable curves on S, continuously differentiable or not. We do not need geometric curvature of any kind. Our paper is:

F. S. Cater "On the curves of minimal length on spheres in real Hilbert spaces," *Real Analysis Exchange* **25**(2), 1999/2000, pp. 781-786.

This work was done for teacher education people in our Department who wanted a proof in 3-space without the use of curvature.

V.4 Open mappings

We use real variable arguments to prove the theorem in complex variables, that the image of a nowhere constant analytic function on an open region is an open set. Such a function must be an open mapping, that is, maps open sets to open sets. Our paper is: F. S. Cater "An elementary proof that analytic functions are open mappings," *Real Analysis Exchange* **27**(1), 2001/2002, pp. 389-392. (One typo: In the statement of Lemma 1, "$\ldots f(0)$ and the set $f(B)\ldots$" should be "$\ldots f(0)$ to the set $f(B)\ldots$". My fault – sorry.)

We used our technique to find yet another proof of the fundamental Theorem of Algebra.

Real variable proofs of complex variable theorems are especially difficult. For our purposes we regarded an analytic function to be a function that coincides locally with the sum of a power series about each point. For more about real analysis proofs of results concerning complex valued functions, consult: F. S. Cater "Another application of Rolle's theorem," *Real Analysis Exchange* **30**(2), 2004/2005, pp. 795-798.

V.5 On finitely generated abelian groups

In "Uniqueness of the decomposition of finite abelian groups: a simple proof," *Mathematics Magazine* **71**(1), February 1998, pp. 50-52, we gave an elementary proof (from first principles) of the uniqueness of the decomposition of finite abelian groups. This can be expressed as follows:

Theorem. Let G be a finite abelian group. Let

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_m = H_1 \oplus H_2 \oplus \cdots \oplus H_n$$

be two decompositions of G; that is, all the G_i and H_i are cyclic subgroups of G, and order G_i divides order G_{i+1} for $1 \leq i \leq m-1$, and order H_i divides order H_{i+1} for $1 \leq i \leq n-1$. Then $n = m$ and order G_i = order H_i for $1 \leq i \leq n$.

The proof usually requires considerable machinery, but we gave a short proof using only first principles. There was one typo: on the last page, "Case 2. Let $r < s$" ... should be "Case 2. Let $r > s$" ... . My fault–sorry.

ADDENDUM.

We prove the following

Lemma. Let K be a torsion free abelian group; that is, every nonzero element in K has infinite order. Let

$$K = K_1 \oplus K_2 \oplus \cdots \oplus K_n = H_1 \oplus H_2 \oplus \cdots \oplus H_m$$

where each K_i and H_i is an infinite cyclic subgroup of K. Then $n = m$.

Proof. We argue by contradiction. Let $n < m$. We will employ vector space theory. Let $v_1, v_2, \ldots, v_n$ be an ordered basis of an n-dimensional vector space V over the field of rational numbers. Let K be the family of vectors (elements) of the form $\sum_{i=1}^{n} a_i v_i$ where each a_i is an integer. Then K is a subgroup of the additive group of V. Moreover K is the direct sum of n infinite cyclic groups of the form $\{av_i\}$ for integers a and each $i = 1, 2, \ldots, n$.

By hypothesis, K is also the direct sum of m infinite cyclic groups $H_1, H_2, \ldots, H_m$. For each $i = 1, 2, \ldots, m$ select a nonzero element u_i in H_i. Then there exist rational numbers $b_1, b_2, \ldots, b_m$, not all zero, such that

$$\sum_{i=1}^{m} b_i u_i = 0 \qquad (1)$$

because m exceeds the dimension of V. Let s be the product of all the integer denominators of the rational numbers b_i. Hence sb_i is an integer for $i = 1, 2 \ldots, m$, not all zero. Add the left side of (1) with itself enough times to obtain

$$s\left(\sum_{i=1}^{m} b_i u_i\right) = \sum_{i=1}^{m} (sb_i) u_i = 0, \qquad (2)$$

not all $sb_i = 0$. But $(sb_i)u_i \in H_i$, and it follows that the sum

$$H_1 + H_2 + \cdots + H_m$$

can not be a direct sum. This contradiction proves $n \geq m$. The proof of $m \geq n$ is analogous.

It remains to prove the uniqueness of the decomposition of finitely generated abelian groups. Let G be a finitely generated abelian group.

By a decomposition of G we mean

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_m,$$

where each G_i is a cyclic subgroup of G, and for each index $i < m$ either order G_i divides order G_{i+1} or G_{i+1} is infinite. Let

$$G = H_1 \oplus H_2 \oplus \cdots \oplus H_n,$$

where each H_i is a cyclic subgroup of G, and for each index $i < n$ either order H_i divides order H_{i+1} or H_{i+1} is infinite. Let s be the largest index for which

G_s is finite and let t be the largest index for which H_t is finite. Let P denote the subgroup composed of all the elements of finite order in G. Clearly

$$P = G_1 \oplus G_2 \oplus \cdots \oplus G_s = H_1 \oplus H_2 \oplus \cdots \oplus H_t$$

and P is a finite group. By the Theorem $s = t$ and order G_i = order H_i for $i = 1, 2, \ldots, s$. For the quotient group G/P it is clear that

$$G/P = G_{s+1} \oplus \cdots \oplus G_m = H_{t+1} \oplus \cdots \oplus H_n,$$

By the Lemma, $m - s = n - t$. But $s = t$, and it follows that $m = n$.

We hope these arguments are of some use in University teaching.

V.6 Salad Days

We get a glimpse of me in my salad days in S. Cater "An elementary development of the Jordan Canonical Form," *Amer. Math. Monthly* **69**(5), May 1962, pp. 391-393.

We gave another argument there in which the underlying field is algebraically closed and we did not use determinants. At the time (circa 1962) it would have been very difficult to find a proof without determinants in print. I suppose this state of affairs is different today.

V.7 On L_p-spaces where $0 < p < 1$

Let p be a real number, $0 < p < 1$, and let L_p denote the family of all functions f on $[0, 1]$ for which $\int_0^1 |f|^p < \infty$, where two such functions are regarded as the same if they differ only on a set of measure zero. Put

$$\zeta(f, g) = \int_0^1 |f - g|^p.$$

It is shown in M. M. Day "The spaces L^p where $0 < p < 1$," **Bull. Amer. Math. Soc.** 46 (1940), pp. 816-823, that L_p is a real topological linear space under the metric ζ that has no nonzero continuous linear functional. The Radon-Nikodym theorem is central to his argument. In S. Cater "Note on a theorem of Day, " *Amer. Math. Monthly* **69**(7), 1962, pp. 638-640, we prove that there is no nonzero continuous linear functional on L_p using first principles without the Radon-Nikodym Theorem.

V.8 On sets where unilateral derivatives are infinite

One of the classical results of real analysis is that the set of points where a real function may have an infinite unilateral derivative necessarily has measure zero. This can be improved as follows.

Theorem. For any real function F, the set of points x at which

$$\lim_{h\to 0^+} |F(x+h)-F(x)|/h = \infty$$

has measure zero.
(Consult for example Chapter IX, Theorem (4.4) of S. Saks, "Theory of the Integral, Second Revised Edition," Dover, New York, 1964.)

However, this reference employs considerable machinery to prove the theorem. We can offer a relatively short proof from first principles. We did not publish it, so we write it here.

We begin with a Lemma that is easy and straight-forward. We provide a proof for completeness.

Lemma. Let F be a function on the compact interval $[a,b]$ and let k be a positive number. Let X be a subset of $[a,b]$ such that for any $x, y \in X$,

$$|F(y)-F(x)|/|y-x| > k.$$

Then $m(F(X)) \geq km(X)$ where m denotes Lebesgue outer measure.

Proof. Let (J_n) denote a sequence of compact intervals that cover $F(X)$: $F(X) \subset \cup_n J_n$.

For a particular index n, let $x, y \in X \cap F^{-1}(J_n)$. Then $|F(x)-F(y)|/k > |x-y|$ and we deduce that

$$\sup(X \cap F^{-1}(J_n)) - \inf(X \cap F^{-1}(J_n)) \leq m(J_n)/k.$$

Define the compact interval

$$I_n = [\inf(X \cap F^{-1}(J_N)), \sup(X \cap F^{-1}(J_n))].$$

It follows that $X \subset \cup_n I_n$ and for any index n,

$$m(I_n) \leq m(J_n)/k.$$

We sum to obtain

$$m(X) \leq \sum_n m(I_n) \leq \sum_n m(J_n)/k.$$

The covering (J_n) of $F(X)$ was arbitrary, so it follows that

$$m(X) \leq m(F(X))/k.$$

Proof of the theorem.

Let X be the set in question and let $m(X) > 0$. There is an interval (c, d) so large that $m(X \cap F^{-1}(c, d)) > 0$. It follows that there is a number $u > 0$ and a subset $X_1 \subset X \cap F^{-1}(c, d)$ such that $m(X_1) > 0$,

$$F(X_1) \subset (c, d) \tag{1}$$

and if $x, y \in X_1$ with $|x - y| < u$, then

$$|F(x) - F(y)| > |x - y| \tag{2}$$

Cover X_1 with finitely many intervals of length $< u$. It follows that there is an interval K with $m(K) < u$ and $m(X_1 \cap K) > 0$.

It follows that there is a number $v > 0$ and a subset $X_2 \subset X_1 \cap K$ such that $m(X_2) > m(X_1 \cap K)/2$ and if $x, y \in X_2$ with $|x - y| < v$, then

$$|F(x) - F(y)|/|x - y| > 8(d - c)/m(X_1 \cap K)$$

and consequently

$$|F(x) - F(y)|/|x - y| > 4(d - c)/m(X_2).$$

Select a finite sequence (I_n) of mutually disjoint compact intervals of length less than v such that

$$\sum_n m(X_2 \cap I_n) > m(X_2)/2. \tag{3}$$

We deduce from (2) that the sets $F(X_2 \cap I_n)$ have mutually disjoint neighborhoods. But if A, B are sets with disjoint neighborhoods, then $m(A \cup B) = m(A) + m(B)$. We deduce that

$$m(\cup_n F(X_2 \cap I_n)) = \sum_n m(F(X_2 \cap I_n)). \tag{4}$$

It follows from our Lemma and the definition of v, that for each index n,

$$m(F(X_2 \cap I_n)) \geq m(X_2 \cap I_n) \cdot 4(d-c)/m(X_2). \tag{5}$$

Thus from (3), (4) and (5) obtains

$$\begin{aligned}
&m(F(X_2)) \\
&\geq m(\cup_n F(X_2 \cap I_n)) = \sum_n m(F(X_2 \cap I_n)) \\
&\geq \sum_n m(X_2 \cap I_n) \cdot 4(d-c)/m(X_2) \\
&= 4(d-c) \sum_n m(X_2 \cap I_n)/m(X_2) \\
&\geq 2(d-c).
\end{aligned}$$

But $X_2 \subset X_1$, and it follows that $m(F(X_1)) \geq 2(d-c)$, contrary to (1).

It is worth noting that F need not be continuous in this argument. We hope that this work is of some use in University teaching.

Chapter VI

Variety

VI.1 Foray into point-set topology

For a topological space X, let $d(X)$ denote the smallest cardinal number that any dense subset of X may have. For cardinal numbers k and λ, let $(X^k)_{(\lambda)}$ denote the λ-box product of k copies of X. For cardinality k, let k^+ denote the smallest cardinal exceeding k, and let cf k denote $\{\min \alpha :$ k is the sum of α cardinals less than $k\}$. Now let $d(X) \geq 2$ and let k and λ be infinite cardinals with $\lambda \leq k^+$. In F. S. Cater, Paul Erdös, Fred Galvin, "On the density of λ-box products," *General topology and Its Applications* 9 (1978), pp. 307-312, we prove the relation cf $(d((X^k)(\lambda))) \geq$ cf λ under this hypothesis.

Initially, I made up some failed problems on products of copies of the doubleton space $\{0, 1\}$ while playing around as I often do. Finally I got it right. I wrote to Erdös who extended it to many more cardinal numbers. We referred it to an expert in point-set topology and set theory who researched the literature, added more material and made it ready for publication. Galvin also used the work to answer a question posed by Comfort and Negrepontis that was open at the time.

So what began as a small matter grew in to something larger. I am happy to have had a hand in all this. In any case, we got a paper in the Journal "Topology and Its Applications," and the Erdös number of two of the co-authors is <u>one</u>.

VI.2 Certain nonconvex linear topological spaces

In S. Cater "On a class of metric linear spaces which are not locally convex," *Math. Annalen* 157 (1964), pp. 210-214, we constructed a nonconvex metric linear space V that enjoys many properties proved for convex metric linear spaces. For example,

1. Every continuous linear functional on V is bounded on V,
2. Any bounded linear functional on any subspace of V can be extended to a bounded linear functional on V.

VI.3 Linear functionals on certain linear topological spaces

In S. Cater "Continuous linear functionals on certain topological vector spaces," *Pacific Journal of Mathematics* **13**(1) (1963), pp. 65-71, we used certain functions ϕ to define certain linear topological spaces V such that continuous linear functionals on V separate points in V if

$$\liminf_{n\to\infty} n^{-1}\phi(n) > 0,$$

and continuous linear functionals on V do not separate points in V if

$$\liminf_{n\to\infty} n^{-1}\phi(n) = 0.$$

VI.4 Foray into fields

If F is a (commutative) field, we let $F(u)$ denote a simple transcendental extension of F. We say that F is an SB field if $F(u)$ is isomorphic to a subfield of F but $F(u)$ is not isomorphic to F. Thus F and $F(u)$ are a pair of nonisomorphic fields, each isomorphic to a subfield of the other. In F. S. Cater "Note on a variation of the Schroeder Bernstein problem for fields," *Czechoslovak Mathematical Journal*, **52**(127) (2002), pp. 717-720, we prove that the field C of complex numbers is an SB field but the field R of real numbers is not. However R contains many subfields that are SB fields.

For authors, we provided a handy reference for the proof that any uncountable algebraically closed field in an SB field. This may have been difficult to find elsewhere in the literature at the time.

VI.5 Collectionwise normal spaces

We provided "A simple proof that a linearly ordered space is hereditarily and completely collectionwise normal," *Rocky Mountain Journal of Mathematics* **36**(4), 2006, pp. 1149-1152,

The central idea I devised for normality around 1955-1956 when I finished college and started graduate school at the University of Southern California. I had no thought of publishing at that time (I was busy making my grades in classes, not all in Mathematics).

They allowed me to give a colloquium lecture at USC on it. I will never forget that curious experience – I may have been the youngest person present. There may also have been some reaction to the fact that I looked even younger than I was.

More recently, I wrote the paper mentioned above for closure.

Compare with Lynn A. Steen, "A direct proof that a linearly ordered space is hereditarily collectionwise normal," *Proc. Amer. Math. Soc.* **24** (1970), pp. 727-728.

VI.6 On real functions of two variables

Everyone should know the standard example of a function of 2 variables with unequal mixed partial derivatives at a point. Put $f(x,y) = xy(x^2-y^2)/(x^2+y^2)$ for $(x,y) \neq (0,0)$, and $f(0,0) = 0$. (See for example, R. Courant, Differential and Integral Calculus, volume II, Blackie & Son, Limited, London, 1936, fine print, p. 57.)

But by a standard theorem, if one of the mixed partial derivatives is continuous at a point, then it must equal the other mixed partial derivative there.

Now we look for global generalizations of this result. Let f be a continuously differentiable function of 2 variables, on R_2, and let the mixed partial f_{xy} exit everywhere. For fixed u and v and positive index n, put

$$F_n(u,v) = n[f_x(u, v+n^{-1}) - f_x(u,v)].$$

Then $F_n(u,v)$ is a continuous function of (u,v), and

$$\lim_{n\to\infty} F_n(u,v) = f_{xy}(u,v).$$

By Osgood's Theorem, it follows that f_{xy} is continuous at each point of a residual subset of the complete metric space R_2. Consult for example, Karl Stromberg, An Introduction to Classical Real Analysis, Wadsworth, Belmont, 1981, p. 120.

By the standard theorem f_{yx} exists and $f_{yx} = f_{xy}$ on a residual set of points. (See Courant, idem. pp. 56-57.) For variations on this global result, see F. S. Cater "Changing the order of partial differentiation," *Real analysis Exchange* **14**(2), (1988-1989) pp. 512-516. Particularly Theorem 3.

VI.7 A variation on T_1-functions

A continuous real valued function f on an interval I is called a T_1-function on I if almost every real value is assumed by f at most a finite number of times. (See S. Saks, Theory of the Integral, Second Revised Edition, Dover, New York 1964. p. 277.)

It follows that the continuous function f is T_1 on I if and only if the set of all values assumed by f at points where f has no derivative, finite or infinite, be a set of measure zero. (See Saks, idem. p. 278.)

One wonders what can be said when f assumes at most countably many values an infinite number of times. In F. S. Cater "Functions that nearly preserve G_δ-sets," *Real Analysis Exchange* **13**(1), 1987-1988, pp. 204-213, we prove that a continuous function f on an interval I assumes at most countably many values infinitely many times on I if and only if for each G_δ-set $S \subset I$, $f(S)$ is the union of a G_δ-set with a countable set.

VI.8 Functions having equal ranges

In Lemmas 1, 2, 3, 4 and 5 of F. S. Cater, "Comparing the ranges of continuous functions," *Real Analysis Exchange* **17**(1), 1991-1992, pp. 426-430, we proved: Let f and g be continuous functions on the interval $[0, 1]$ with $f(0) = g(0) = 0$. For each subinterval I of $[0, 1]$, let the intervals $f(I)$ and $g(I)$ have the same length. Then either $f + g$ is identically zero on $[0, 1]$ or $f - g$ is identically zero on $[0, 1]$.

At first I did not believe this, and I was looking for a counterexample. But when I could not find one I became suspicious.

Many direct assaults on the proof of this result turn out to be faulty. I believe it is harder than it looks.

VI.9 Foray into ring theory

We say that a commutative ring R is an Artinian ring if any contracting sequence of ideals $I_1 \supset I_2 \subset I_3 \supset \ldots$ must terminate: that is, $I_k = I_{k+1} = I_{k+2} = \ldots$ for some index k. We say that a commutative ring R is almost Artinian if for some index k,

$$R^k I_k \subset \cap_n I_n.$$

Of course R is almost Artinian if R is Artinian.

If R is a commutative ring with identity, then R is an Artinian ring if and only if R is an almost Artinian ring. Clearly a nilpotent commutative ring must be an almost Artinian ring.

In theorem 3 of F. S. Cater, "Modified chain conditions for rings without identity," *The Yokohama Mathematical Journal*, vol XXVII, no. 1, 1979, pp. 1-22, we proved that a commutative ring R is an almost Artinian ring if and only if R is the direct sum of an Artinian ring with identity and a nilpotent ring.

We also discussed related results for noncommutative rings there.

VI.10 Mappings into sets of measure zero

Let g be a function of bounded variation on $[0, 1]$ and for any indices i and n, $0 < i \leq 2^n$, let J_{in} denote the interval

$$J_{in} = [(i-1)2^{-n}, i2^{-n}].$$

Let G denote the total variation function, $G(u) =$ total variation of $g(x)$ on the interval $[0, u]$.

In Lemma 2 of F. S. Cater, "Mappings into sets of measure zero," *Rocky Mountain Journal of Mathematics* **16**(1), Winter, 1986,
we proved that

$$m(G(S)) = \lim_{n\to\infty} \sum_{i=1}^{2^n} m(g(J_{in} \cap S))$$

where m denotes Lebeague outer measure and S is any subset of $[0, 1]$.

Elsewhere in the paper we made some general comments about N-functions, absolutely continuous functions, singular functions and saltus functions.

While an undergraduate student in the early 1950s I began studying saltus functions on my own, a manifestation of my early interest in real variables.

VI.11 On upper and lower integrals

In F. S. Cater "Upper and lower generalized Riemann integrals," *Real Analysis Exchange* **16** (1990-1991), pp. 215-237, we defined upper and lower integrals that serve much the same purpose for the Henstock integral that the Darboux upper and lower integrals serve for the Riemann integral.

In the mid 1950s when I was a student, I was trying to define an integral with many of the properties of the Henstock integral, but nothing seemed to work. I played with it for some time, but the essential notion of a "gauge" eluded me. Needless to say, I regret missing it.

VI.12 On closed subsets of uncountable closed sets

In this Item we prove the following Theorem.

Theorem. Let X be an uncountable complete separable metric space. Then X has an uncountable complete subset W enjoying the following property: For any $\epsilon > 0$, W can be covered by finitely many balls the sum of whose radii is less than ϵ.

A proof of our Theorem could be messy, but we will provide a systematic solution. The trouble is two-fold. If we select the points in W such that the Property is satisfied, will W be an uncountable complete set? But if we construct W by discarding points in X to form a set large enough to be an uncountable complete set, will W satisfy the Property?

Proof of the Theorem.

Let X_0 denote the set of points x in X every neighborhood of which meets X in uncountably many points. Clearly X_0 is a closed subset of X, and hence is a complete subset of X. Now X is a separable metric space and therefore has a countable basis. It follows that the set $X - X_0$ is an open countable subset of X, so X_0 is an uncountable set. We can (and do) assume, without loss of generality, that X has no isolated point.

Let S denote the family of all finite sequences of 1s and 2s. For $s \in S$ let "length s" denote the number of entries in s. We will use inductive construction on length s to define a closed ball U_s in X centered at a point x_s in X having radius less than $3^{-(\text{length})}$ as follows.

Choose disjoint closed balls U_1 and U_2 centered at points $x_!$ and x_2 which have radius less than $1/3$. Now assume that mutually distinct closed balls U_s have been selected for all sequences s in S of length $\leq w$. Let s be a sequence of length w. Choose disjoint closed balls U_{s1} and U_{s2} of radii less than 3^{-w-1} with $U_{s1} \subset U_s$ and $U_{s2} \subset U_s$. Let x_{s1} and x_{s2} denote the respective centers of the balls U_{s1} and U_{s2}.

By inductive construction all the balls with subscripts in S have been chosen. For each n let V_n denote the union of all the balls U that have subscripts of length n. Then V_n is a closed subset of X.

Now let t be an infinite sequence of 1s and 2s. Associate with t the Cauchy sequence (y_n) where y_n denotes the center of the ball whose subscript is the first n entries of t. But X is complete, so (y_n) must converge.

Let w_t denote the limit of (y_n). By construction, w_t must lie in the balls just mentioned, and

$$w_t \in \cap_n V_n.$$

For any two different sequences t_1 and t_2, the limits w_{t_1} and w_{t_2} lie in disjoint closed balls and $w_{t_1} \neq w_{t_2}$. Thus $\cap_n V_n$ contains at least as many points as there are sequences t. It follows that $\cap_n V_n$ is uncountable. Put

$$W = \cap_n V_n.$$

For any index n, V_n is the union of 2^n balls (covering W) each of whose radius is less than 3^{-n}. The sum of these radii is less than

$$2^n(3^{-n}) = (2/3)^n.$$

Of course W is complete because each V_n is closed and X is complete. □

We conclude with a Corollary that was mentioned for R back in Item III(6).

Corollary. Let X be a closed uncountable subset of the Euclidean space R_k $(k \geq 1)$. Then X contains a closed uncountable subset W with measure zero.

Proof. Any ball of radius r in R_k can be enclosed in a box with edge $2r$. The measure of this box is $(2r)^k$. Let $0 < \epsilon < 1$. In our Theorem, W was covered with balls the sum of whose radii was less than ϵ. It follows that $m(W) < \sum_j 2^k {r_j}^k < 2^k (\sum r_j)^k < 2^k \epsilon^k < 2^k \epsilon$.

But ϵ was arbitrary, so $m(W) = 0$. □

Postscript

In recent decades there has been a resurgence of research in what is called “real variables”, or “real functions” or “real analysis.” That this field is massive is suggested, for example, by the book “Real functions – current topics” by Vasily Ene, Springer, New York, 1995. That this field is active is suggested by issues of the Journal, “Real analysis Exchange”, Michigan State University Press.

My work has been primarily in real variables, though not entirely. I did think of a few other things.

You may have noticed there was an emphasis here on nowhere differentiable functions. This is appropriate because my academic lineage goes back to Karl Weierstrass.