IMPROPER LEBESGUE INTEGRABLE FUNCTIONS WHICH ARE NOT RANDOM RIEMANN INTEGRABLE

Abstract

We give two conditions which are sufficient for a Henstock-Kurzweil integrable function on the unit interval to not be random Riemann integrable. This also extends the class of functions which are not almost sure first-return integrable. One condition requires that the function be bounded on any closed set which does not include 0. The other condition is on the size of the sets \( \{ x : f(x) > y \} \), and means that the size of these sets is asymptotically larger than it would be for the function \( f(x) = \frac{1}{x} \).

In the following each function is Lebesgue measurable and from the unit interval \([0, 1]\) to the real line. We adopt some more-or-less standard notation from [3].

Let \( f \) be a function as above and \( \mathcal{P} \) be a partition of \([0, 1]\), that is to say, \( \mathcal{P} \) is a finite collection of intervals which are pairwise non-overlapping and whose union is \([0, 1]\).

Definition 1. The \textit{random Riemann sum of } \( f \) \textit{on } \( \mathcal{P} \) \textit{is the random variable}

\[
S_{\mathcal{P}}(f) := \sum_{I_k \in \mathcal{P}} |I_k| f(t_k)
\]

where, for each \( I_k \), \( t_k \) is a random variable distributed on \( I_k \) uniformly and independently of all other \( t_i \).

Since \( f \) is Lebesgue measurable, \( S_{\mathcal{P}}(f) \) is also Lebesgue measurable.

Definition 2. A function is \textit{random Riemann integrable} and \( M \) is its \textit{random Riemann integral} iff for every \( \varepsilon > 0 \) there exists \( \delta \) such that

\[
|\mathcal{P}| < \delta \implies P(|S_{\mathcal{P}}(f) - M| > \varepsilon) < \varepsilon.
\]
In [3] the following theorem was proved.

**Theorem 3.** If \( f \) is Lebesgue integrable, then it is random Riemann integrable and the values of the two integrals are the same.

It has been possible to show a partial converse: that a restricted class of Henstock-Kurzweil integrable functions which are not Lebesgue integrable, are also not random Riemann integrable. We give sufficient conditions for this to be the case.

**Condition A.** For every \( \delta > 0 \), \( f \) is bounded on \([\delta, 1]\).

This implies that \( f \) is not just Henstock-Kurzweil integrable, but also improper Lebesgue integrable, with the Henstock-Kurzweil integral equal to the limit

\[
\lim_{\delta \to 0} L \int_{\delta}^{1} f.
\]

It is of course enough to replace ‘bounded’ by ‘essentially bounded’.

**Condition B.** There exists a constant \( A \), depending only on \( f \), such that for each \( z \) there exists \( w > z \) for which

\[
|\{ x : f(x) > w \}| > \max \left( A/w, \frac{1}{2} \left| \left\{ x : f(x) > \frac{w}{2} \right\} \right| \right) \quad (1)
\]

holds.

This condition provides a lower bound on the size of the portion of \( f \) greater than some value. It is straightforward to show that if a function is not in the class \( L_{1-\varepsilon} \) for some \( \varepsilon > 0 \), then condition B must hold.

We can now state the main theorem.

**Theorem 4.** Suppose that a function \( f \) satisfies conditions A and B. Then \( f \) is not random Riemann integrable.

Of course we can equally well prove the theorem in the case where condition B holds for \(-f\), since condition A and the property of random Riemann integrability are unchanged under multiplication of \( f \) by \(-1\).

Random Riemann integrability is a weaker condition than the following two properties of real-valued functions on intervals, to which it is related.

**Definition 5.** The space of sequences of points in \([0, 1]\) together with the infinite product probability measure of the uniform distribution on \([0, 1]\), is denoted \( \Omega \).
Definition 6. A function $f$ is almost sure first-return integrable iff for almost every sequence $\bar{x}$ in $\Omega$, it is first-return integrable w.r.t. $\bar{x}$, in the sense given in [2].

Definition 7. A function $f$ is almost sure random Riemann integrable iff for every sequence of partitions $P_n$ with size tending to 0, the sequence of random variables $S_{P_n}(f)$ tends to a limit almost surely.

Every function which is almost sure random Riemann integrable is almost sure first-return integrable. We can therefore obtain the following consequence of Theorem 4.

Corollary 8. Suppose $f$ satisfies conditions A and B. Then $f$ is not almost sure first-return integrable, neither is it almost sure random Riemann integrable.

This theorem extends the ‘Class B’ family of not almost sure first-return integrable functions defined in [1], since some functions which are not piecewise either positive or negative satisfy these two conditions.

It should be observed that no functions (other than those which are Lebesgue equivalent to Riemann integrable function, for which the properties hold trivially) have been shown to be either a.s. first-return integrable or a.s. random Riemann integrable. All positive results about either of these two properties concern weakened versions of the definitions, in which the set of possible sequences of partitions is restricted.

It remains to offer a conjecture concerning random Riemann integrability.

Conjecture 9. A function is random Riemann integrable exactly iff it is Lebesgue integrable.

A possible first step towards proving this conjecture might be to attempt to show that a function for which either condition A or condition B but not necessarily both holds, is not random Riemann integrable.

References

