DISTRIBUTIONAL CHAOS AND IRREGULAR RECURRENCE

For a continuous map \( \varphi : X \to X \) of a compact metric space, we study relations between distributional chaos and the existence of a point which is quasi-weakly almost periodic, but not weakly almost periodic. We provide an example showing that the existence of such a point does not imply the strongest version of distributional chaos, \( \text{DC1} \). Using this we prove that, even in the class of triangular maps of the square, there are no relations to \( \text{DC1} \). This result, among others, contributes to the solution of a problem formulated by A.N. Sharkovsky in the eighties.

1 Introduction.

Let \((X, \rho)\) be a compact metric space, and \( \varphi : X \to X \) a continuous map. By \( \omega_\varphi(x) \) we denote the \( \omega \)-limit set of \( x \) which is the set of limit points of the trajectory \( \{ \varphi^i(x) \}_{i \geq 0} \) of \( x \) where \( \varphi^i \) denotes the \( i \)th iterate of \( \varphi \). For \( x \in X \) and \( U \) a neighborhood of \( x \), let

\[
P_x(U) = \liminf_{n \to \infty} \frac{1}{n} \# \{ 0 \leq j < n; \varphi^j(x) \in U \},
\]

\[
\overline{P}_x(U) = \limsup_{n \to \infty} \frac{1}{n} \# \{ 0 \leq j < n; \varphi^j(x) \in U \}.
\]

A point \( x \in X \) is quasi-weakly almost periodic with respect to \( \varphi \) or, \( x \in \text{QW}(\varphi) \) if, for every neighborhood \( U \) of \( x \), \( P_x(U) > 0 \). It is weakly almost periodic or, \( x \in \text{W}(\varphi) \) if for any neighborhood \( U \) of \( x \), \( \overline{P}_x(U) > 0 \). Obviously, \( \text{W}(\varphi) \subseteq \text{QW}(\varphi) \). The properties of \( \text{W}(\varphi) \) and \( \text{QW}(\varphi) \) were studied in the nineties in several papers by Z. Zhou et al, see [11] for references. In particular, any minimal set of \( \varphi \) (i.e., a closed set \( M \neq \emptyset \) such that \( \varphi(M) = M \), and no proper subset of \( M \) has these properties) is contained in \( \text{W}(\varphi) \). The points

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Mathematical Reviews subject classification: Primary: 37B20, 37D45; Secondary: 37B40
in $IR(\varphi) = QW(\varphi) \setminus W(\varphi)$ are irregularly recurrent points; i.e. points $x$ such that $P_x(U) > 0$ for any neighborhood $U$ of $x$, and $P_x(V) = 0$ for some neighborhood $V$ of $x$.

Recall that two points $x, y \in X$ are proximal if $\lim \inf_{n \to \infty} \rho(\varphi^n(x), \varphi^n(y)) = 0$; they are asymptotic if $\lim_{n \to \infty} \rho(\varphi^n(x), \varphi^n(y)) = 0$. A system $(X, \varphi)$ is Li-Yorke chaotic, or LYC, if $X$ contains a pair of proximal points which are not asymptotic. Finally, we recall the notion of distributional chaos. For any pair $(x,y)$ of points in $X$, and any $t$ with $0 < t \leq \text{diam}(X)$ put $d_{xy}(j) = \rho(\varphi^j(x), \varphi^j(y))$, and let

$$\Phi_{xy}(t) = \lim \inf_{n \to \infty} \frac{1}{n} \# \{0 \leq j < n; d_{xy}(j) < t\},$$

$$\Phi_{xy}^*(t) = \lim \sup_{n \to \infty} \frac{1}{n} \# \{0 \leq j < n; d_{xy}(j) < t\}.$$

Then $\Phi_{xy}$ and $\Phi_{xy}^*$ are the lower or the upper distribution functions of $x$ and $y$, respectively. Obviously, $\Phi_{xy}(t) \leq \Phi_{xy}^*(t)$ for any $t \in (0, \text{diam}(X)]$. If $\Phi_{xy}(t) \leq \Phi_{xy}^*(t)$ for all $t$ in an interval, we write $\Phi_{xy} \leq \Phi_{xy}^*$. There are three types of distributional chaos, $DC1 – DC3$, which are given by the following conditions:

$$\Phi_{xy}^* \equiv 1 \text{ and } \Phi_{xy}(t) = 0, \text{ for some } x, y \in X \text{ and some } t > 0, \quad (DC1)$$

$$\Phi_{xy}^* \equiv 1 \text{ and } \Phi_{xy}(t) < 1, \text{ for some } x, y \in X \text{ and some } t > 0, \quad (DC2)$$

while $DC3$ means that there are points $x, y \in X$ such that $\Phi_{xy} < \Phi_{xy}^*$. The strongest type of chaos, $DC1$, was originally introduced in [9], $DC2$ and $DC3$ are generalizations [10]. Recall that, in general, these notions are mutually not equivalent, with $DC1 \Rightarrow DC2 \Rightarrow DC3$, and $DC2 \Rightarrow LYC$, cf. [2]. Moreover, $IR(\varphi) \neq \emptyset$ implies $LYC$ (cf. Section 2).

In this paper we are concerned with the problem, how (and if) the existence of irregularly recurrent points is related to distributional chaos. It turns out that the situation is not so nice as with $LYC$. In Section 2 we show that if $(X, \varphi)$ is an almost one-to-one extension of an adding machine then $\varphi$ cannot be $DC1$. Then, using this result, we provide an example of a map possessing an irregularly recurrent point, which is not $DC1$. In the last section we provide other examples, relations to other properties, and some open problems. In particular, we show that even in the class of triangular maps, $DC1$ neither implies nor is implied by the existence of an irregularly recurrent point. This contributes to the problem of classification of triangular maps, which has been formulated in the eighties by A.N. Sharkovsky and which still is not solved. For related results, concerning distributional chaos see, e.g., the recent papers [6] – [8] where other references can be found.
To fix the terminology, note that a skew-product map \( X \times Y \to X \times Y \), where \( X, Y \) are compact metric spaces, is a map \( F : (x, y) \mapsto (f(x), g(x, y)) \) continuous with respect to the max-metric on \( X \times Y \). In the particular case when \( X = Y \) is the unit interval \( I = [0, 1] \), \( F \) is a triangular map; we let \( T \) denote the class of triangular maps. Other notions and terminology are introduced later in this text.

2 Existence of Irregularly Recurrent Point Does Not Imply DC1.

**Theorem 1.** Let \( \varphi \) be a continuous map of a compact metric space \( X \) such that \( IR(\varphi) \neq \emptyset \). Then \( \varphi \) is LYC.

**Proof.** Let \( x \in IR(\varphi) \). By the well-known result by Auslander [1] and Ellis [4], any point in \( X \) is proximal to a point in a minimal set. So, let \( M \subset X \) be a minimal set and \( y \in M \) a minimal point proximal to \( x \). Since \( x \notin W(\varphi) \), and since \( M \subset W(\varphi), x \neq y \). Finally, since \( M = \varphi(M) \) is closed and invariant and \( x \) is recurrent, the points \( x, y \) are not asymptotic. Thus, \( \varphi \) is LYC. \( \square \)

**Theorem 2.** There is a skew-product map \( F \) of the space \( C \times I \), where \( C \) is the Cantor set and \( I = [0, 1] \) the unit interval such that

(i) \( IR(F) \neq \emptyset \);

(ii) \( F \) is DC2 but not DC1;

(iii) \( F \) has zero topological entropy.

**Sketch of the Proof.** I recall that an adding machine associate to a sequence \( \{n_j\}_{j=1}^\infty \) of integers \( n_j > 1 \) is a Cantor-type set \( Q \subset I \) and a continuous map \( \tau \) of \( Q \) such that, for any \( k \geq 1 \), \( Q \) has a decomposition into \( p_k = n_1 n_2 \cdots n_k \) periodic portions forming a unique \( \tau \)-periodic orbit. It is assumed that any two distinct points of \( Q \) belong to distinct periodic portions of period \( p_k \), for some \( k \). Any odometer is a minimal system. A system \( (X_2, \varphi_2) \) is an extension of \( (X_1, \varphi_1) \) if there is a continuous surjective map \( \psi : X_2 \to X_1 \) called a factor map such that \( \psi \circ \varphi_2 = \varphi_1 \circ \psi \). This extension is almost one-to-one if the set of \( x \in X_1 \) with \( \#\psi^{-1}(x) = 1 \) is dense in \( X_1 \).

In the proof we use the following Lemma and Theorem:

**Lemma 1.** If \((X, \varphi)\) is an almost one-to-one extension of an adding machine, then for any \( \varepsilon > 0 \) there is a clopen set \( G \subset X \) with \( \text{diam}(G) < \varepsilon \), and an integer \( m > 0 \) such that \( \varphi^m(G) \subseteq G \).
**Theorem 3.** If \((X, \varphi)\) is an almost one-to-one extension of an adding machine, then \(\varphi\) cannot be \(DC1\).

\[\square\]

3 Concluding remarks

For a continuous map \(\varphi\) of the interval, there are more than 50 conditions equivalent to positive topological entropy, like \(DC1 - DC3, IR(\varphi) \neq \emptyset\), existence of a period \(\neq 2^n\), existence of a homoclinic trajectory, etc. Some of them are applicable to maps in \(T\); there were considered 23 such conditions which are mutually nonequivalent. It is proven that there are no implications between \(DC1\) and the other conditions, except that existence of a homoclinic trajectory, or existence of a period \(\neq 2^n, n \in \mathbb{N}\), implies \(DC1\) [8] and, by definition, \(DC1 \Rightarrow DC2 \Rightarrow DC3\). The next theorem shows that the list of conditions can be extended by another one.

**Theorem 4.** For any \(F \in T\), the properties \(DC1\) and \(IR(F) \neq \emptyset\) are independent; i.e., there is no implication between them.

**Proof.** By Theorem 2 there is a skew-product map \(F : C \times I \to C \times I\) which is not \(DC1\) and has an irregularly recurrent point. This map can be extended to a triangular map using the convex combinations of fibre maps: if \(x\) is an interval \((a, b)\) complementary to \(C\) let \(t \in (0, 1)\) be such that \(x = ta + (1-t)b\); then we let \(g(x, y) = t \cdot g(a, y) + (1-t) \cdot g(b, y)\). The extended map remains to be nondecreasing on the fibres so it has zero topological entropy since \(\tau\) has zero topological entropy; see, e.g., [5] or [10]. Hence, in \(T\), existence of an irregularly recurrent point does not imply \(DC1\). The remainder of the proof follows by the next lemma.

**Lemma 2.** There is a \(DC1\) map \(F \in T\) such that \(QW(F) = W(F)\) is the set of fixed points of \(F\).

**Proof.** The idea is from [5] where there is a map \(F \in T, F(x, y) = (f(x), g(x, y))\) such that every recurrent point is fixed, and the only infinite \(\omega\)-limit set of \(F\) is the set \(I_0 := \{0\} \times I\) of fixed points. So, for any \(x, y \in I\), we let \(f(x) = x/2\) and \(g(0, y) = y\). Then, no matter how the functions \(g(x, y)\) are defined (under the condition that \(F\) is continuous), \(I_0\) is the set of recurrent points of \(F\) and since it consists of fixed points, we have \(QW(F) = W(F) = I_0\).

Let \(x_j = 2^{-j}, j \in \mathbb{N}\). First we define the maps \(g(x_j, \cdot)\) to obtain (a continuous) \(DC1\) skew-product map of \((\{0\} \cup \{x_j\}_{j \geq 1}) \times I\). For any \(j \in \mathbb{N}\), let \(\psi_j : I \to I\) be a map \(y \mapsto \lambda_j y\), with \(0 < \lambda_1 < \lambda_2 < \cdots < 1\), and let \(\vartheta_j\) be a left inverse to \(\psi_j\). Let \(n_0 = 0 < n_1 < n_2 < \cdots\) be integers with
\[ \lim_{j \to \infty} n_{j+1}/n_j = \infty \text{ and } \lim_{k \to \infty} \lambda_k^{n_{3k} - n_{3k-1}} = 0. \]

Put \( g(x_j, y) = \psi_k(y) \) if \( n_{3k-1} < j \leq n_{3k} \), \( g(x_j, y) = \vartheta_k(y) \) if \( n_{3k+1} < j \leq n_{3k+1} + n_{3k} - n_{3k-1} \), and \( g(x_j, y) = y \) otherwise, for \( k \in \mathbb{N} \). It is easy to see that the map is continuous and, for \( u = (x_1, 0), v = (x_1, 1) \), \( \Phi_{uv}^* \equiv 1 \) and \( \Phi_{uv} \equiv 0 \). To finish the argument, it suffices to extend this map to a triangular map as in the proof of Theorem 3.

We conclude with some remarks. First we recall the open and probably difficult problem whether, for a continuous map \( \varphi \) of a compact metrisable \( X \), positive topological entropy (briefly, PTE) implies DC2 or, at least, DC3? From Theorem 3 it follows that \( IR(\varphi) \neq \emptyset \) does not imply PTE, even for maps in \( T \), see Theorem 4 and its proof. The converse implication obviously is not true when \( X \) is a minimal set with PTE. It would be interesting to know whether this implication is true, e.g., in \( T \). Finally, we believe that \( IR(\varphi) \neq \emptyset \) along with PTE imply DC2.

Acknowledgments

The research was supported, in part, by project MSM4781305904 from the Czech Ministry of Education. The author also thanks Professor Jaroslav Smítal for his guidance and suggestions.

References


